Grade 6 Math Circles
March 1st, 2022
Circles Extension

Introduction

We looked at circles and 3D shapes with circles last week. Today, we will be learning more properties about circles. In particular, we will learn about sectors of circles, which can help us explain how we got the area of the side of the cone.

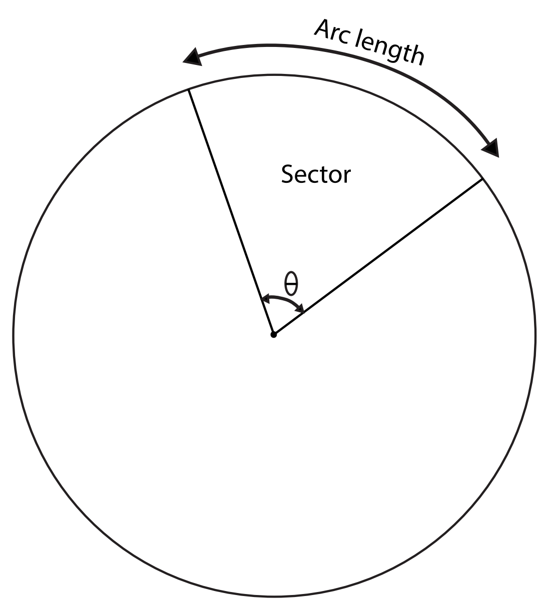
Sectors of Circles

To begin, we can introduce some terminology.

A **sector** of a circle is similar to a slice of pie, where a triangular portion of the circle is sliced from the centre of the circle to the edge of the circle.

Arc length is the length of part of the edge of a circle, for example, the length of the outer edge of a section.

An **angle** is a measurement in degrees of a corner between two lines. An entire circle is 360°, and we can represent angles with the special symbol θ, spelled as “theta” (pronounced as thay-ta).



We can calculate the area of a sector in multiple ways.

1) If we know the fraction of the circle that a sector is, we can simply multiply the area of the circle by the fraction. For example, if the sector is $\frac{1}{4}$ of the circle, then the area of the sector will clearly be $\frac{1}{4}$ of the area of the circle.

2) If we know the angle of the sector, we can use the formula

$$A = \frac{\theta}{360^\circ} \times \pi r^2$$

Breaking this formula down, it's creating a ratio of the angle of the sector and the total angle of the circle, then multiplying that ratio by the area of the circle.

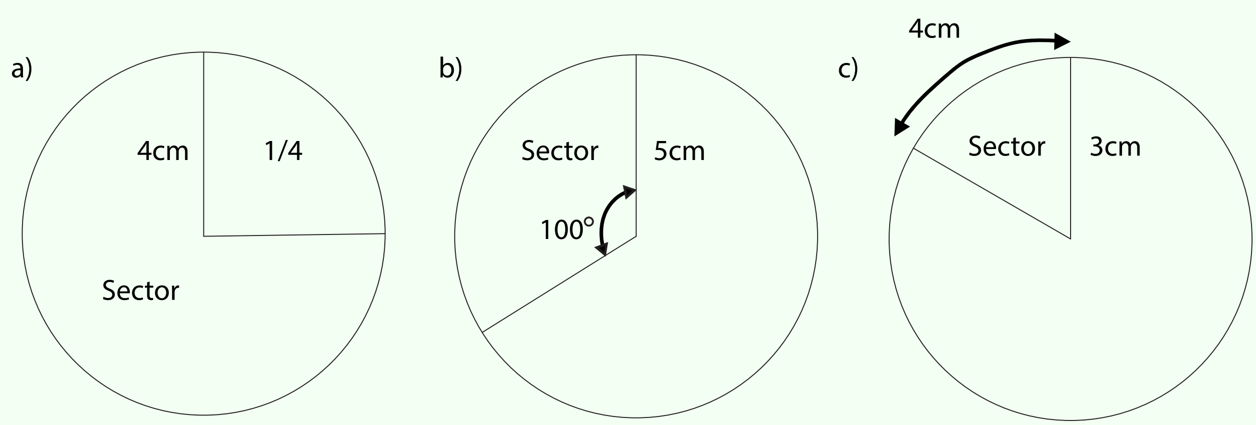
3) If we know the arc length of the sector, we can use the formula

$$A = \frac{\text{Arc length}}{2\pi r} \times \pi r^2$$

Once again, we are creating a ratio but this time with the arc length and the circumference of the circle, then multiplying it by the area of the circle.

Example 1

Calculate the area of each sector below. Round to one decimal point.



a) We have that $r = 4$cm and that the sector is $\frac{3}{4}$ of the circle. We can also use 3.14 instead of π due to only needing one decimal point of accuracy. We can simply calculate the area of the whole circle multiplied by $\frac{3}{4}$ to get the area of the sector:

$$\begin{aligned}\text{Area of sector} &= 3.14 \times (4\text{cm})^2 \times \frac{3}{4} \\ &= 3.14 \times (4\text{cm} \times 4\text{cm}) \times \frac{3}{4} \\ &= 3.14 \times 16\text{cm}^2 \times \frac{3}{4} \\ &= 37.68\text{cm}^2\end{aligned}$$

So the area of this sector, rounded to one decimal point, is 37.7cm^2.

b) We have that $r = 5$cm and that $\theta = 100°$. Using the formula from method two of calculating area of sectors, we can calculate:

$$\begin{aligned}\text{Area of sector} &= \frac{100°}{360°} \times 3.14 \times (5\text{cm})^2 \\ &= \frac{100°}{360°} \times 3.14 \times (5\text{cm} \times 5\text{cm}) \\ &= \frac{100°}{360°} \times 3.14 \times 25\text{cm}^2 \\ &= 21.8055555...\text{cm}^2\end{aligned}$$

So the area of this sector, rounded to one decimal point, is 21.8cm^2.

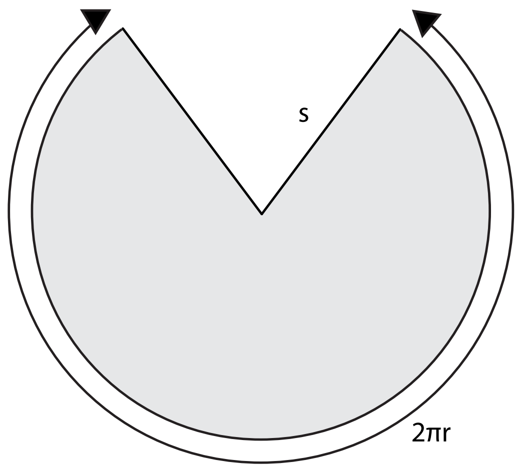
c) We have that $r = 3$cm and that the arc length is 4cm. Using the formula from method three of calculating area of sectors, we can calculate:

$$\begin{aligned}\text{Area of sector} &= \frac{4\text{cm}}{2 \times 3.14 \times 3\text{cm}} \times 3.14 \times (3\text{cm})^2 \\ &= \frac{4\text{cm}}{2 \times 3.14 \times 3\text{cm}} \times 3.14 \times (3\text{cm} \times 3\text{cm}) \\ &= \frac{4\text{cm}}{18.84\text{cm}} \times 3.14 \times 9\text{cm}^2 \\ &= 6\text{cm}^2\end{aligned}$$

So the area of this sector, rounded to one decimal point, is 6.0cm^2.

Looking back at the formula for surface area of a cone, we can now figure out why the area of the side of the cone is equal to $\pi r s$.

When we look at the side flattened out, as mentioned in the lesson, it looks like the following diagram. The arc length of the cutout is equal to $2\pi r$ because that's the circumference of the original cone. Then, the circle that the cutout belongs to would have a circumference of $2\pi s$ because s is the radius of the cutout.

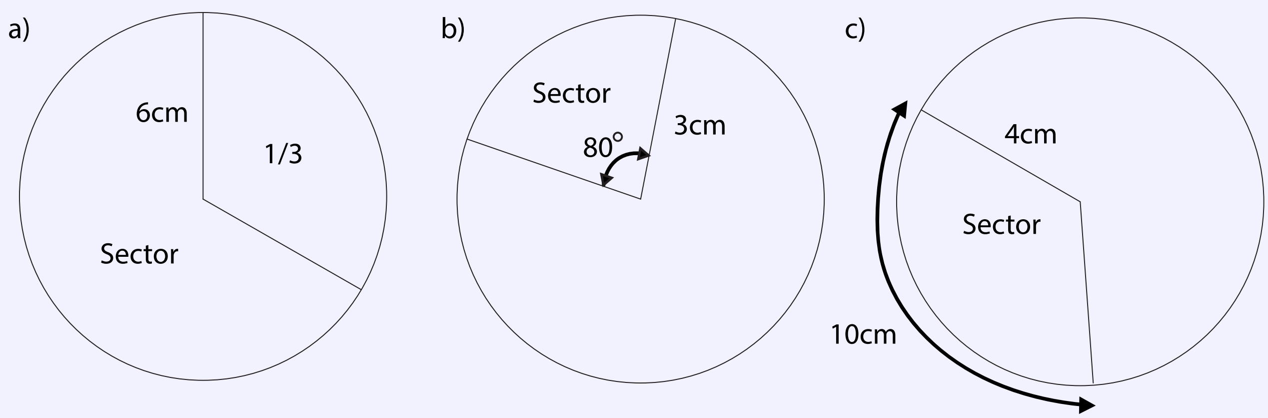


Therefore, using method three for calculating area of a section, we can do some algebra and see that

$$\begin{aligned}\text{Area of cutout} &= \frac{2\pi r}{2\pi s} \times \pi s^2 \\ &= \frac{r}{s} \times \pi s^2 \\ &= \pi \times r \times \frac{1}{s} \times s^2 \\ &= \pi \times r \times s\end{aligned}$$

Exercise 1

Calculate the area of each sector below. Round to one decimal place.



Exercise 1 Solution

We can use 3.14 instead of π for each part since we only need one decimal point of accuracy.

a) We can use method one for this problem. We have that $r = 6\text{cm}$. And, since the sector is $\frac{2}{3}$ of the circle, we can multiply the area of the whole circle by $\frac{2}{3}$ to get our answer:

$$\begin{aligned}\text{Area of sector} &= 3.14 \times (6\text{cm})^2 \times \frac{2}{3} \\ &= 3.14 \times 36\text{cm}^2 \times \frac{2}{3} \\ &= 75.36\text{cm}^2\end{aligned}$$

So the area of this sector, rounded to one decimal point, is 75.4cm^2.

b) We can use method two for this problem. We have that $r = 3\text{cm}$ and $\theta = 80°$, therefore we can calculate:

$$\begin{aligned}\text{Area of sector} &= \frac{80°}{360°} \times 3.14 \times (3\text{cm})^2 \\ &= \frac{80°}{360°} \times 3.14 \times 9\text{cm}^2 \\ &= 6.28\text{cm}^2\end{aligned}$$

So the area of this sector, rounded to one decimal point, is 6.3cm^2.

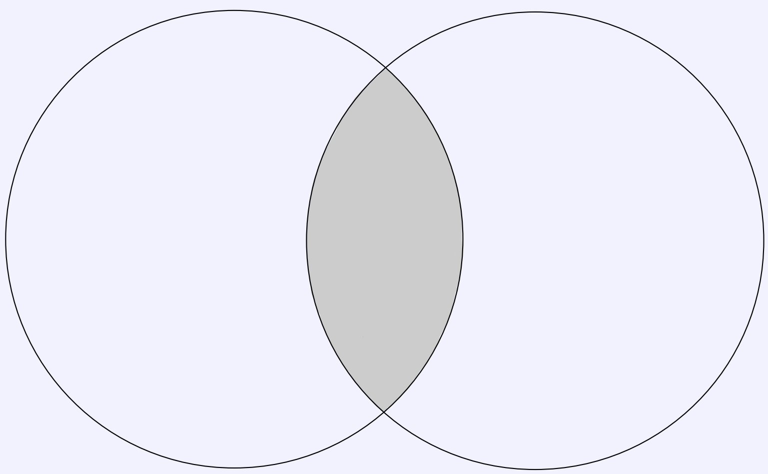
c) We can use method three for this problem. We have that $r = 4$cm and that the arc length is 10cm. Therefore, we can calculate:

$$\begin{aligned}\text{Area of sector} &= \frac{10\text{cm}}{2 \times 3.14 \times 4\text{cm}} \times 3.14 \times (4\text{cm})^2 \\ &= \frac{10\text{cm}}{25.12\text{cm}} \times 3.14 \times 16\text{cm}^2 \\ &= 20\text{cm}^2\end{aligned}$$

So, the area of this sector, rounded to one decimal point, is 20.0cm^2.

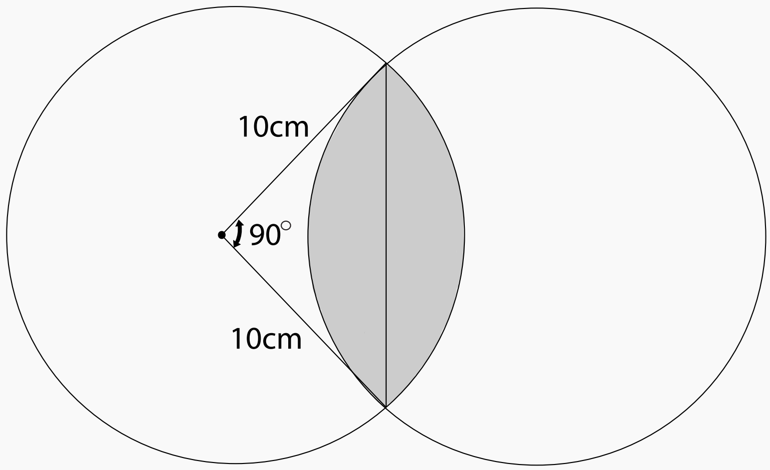
Exercise 2

The two circles below each have a radius of 10cm. They overlap so that each contains exactly 25% of the other's circumference, as shown. What is the area of the shaded region, rounded to one decimal point?



Exercise 2 Solution

We can solve this problem using sectors! First, we should create a diagram to help us understand the problem. We draw the sector corresponding to where the circles overlap, and also creating a triangle will help us later. The left corner of this triangle is a 90° angle because the sector is $\frac{1}{4}$ of the circle. Furthermore, since the radius of each circle is 10cm, we get that the base and height of the triangle are both 10cm.



Now, we need to know that the formula for area of a triangle is $A = \frac{1}{2} \times base \times height$. This will help us, because if we take the area of the sector and subtract the area of our triangle, we are left with the area of half of the shaded region. Then, we can simply multiply that by two to get the full area of the shaded region!

Let's calculate the area of the sector first. We know that the sector is $\frac{1}{4}$ of the circle, so using method one, we get that

$$\begin{aligned}\text{Area of sector} &= 3.14 \times (10\text{cm})^2 \times \frac{1}{4} \\ &= 3.14 \times 100\text{cm}^2 \times \frac{1}{4} \\ &= 78.5\text{cm}^2\end{aligned}$$

Now, we can calculate the area of the triangle using the formula mentioned earlier and get that

$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2} \times 10\text{cm} \times 10\text{cm} \\ &= 50\text{cm}^2\end{aligned}$$

So, the area of the right half of the shaded region must be equal to $78.5\text{cm}^2 - 50\text{cm}^2 = 28.5\text{cm}^2$. Therefore, the area of the whole shaded region must be $28.5\text{cm}^2 \times 2 = 57.0\text{cm}^2$.