Grade 6 Math Circles

March 9, 2022

The A, B, Γ's of Math - Problem Set

1. As a mathematician, you observe that there are some Greek letters that you just never encounter unless you work in very specific fields of math. The letters ι (iota), o (omicron), ν (nu), and υ (upsilon) are some examples of uncommonly used Greek letters.
   (a) Why might these Greek letters not be used as variables?
   (b) What letters in the English alphabet would you not expect to be used as variables?
2. Solve for the unknown variables.
   (a) A circle has a diameter δ (delta). Given that the circumference of the circle is $C = 10\tau$ units, find the value of δ.
   (b) A circle has a radius ρ. Given that the area of the circle is $A = 25\pi$ square units, find the value of ρ.
   (c) **Challenge:** A right-angle triangle Δ has angles α, β and θ. Given that $\theta = 2\alpha$, what are some possible values of α, β, and θ? There is more than one possible answer.
3. Recall that the prime-counting function, $\pi(x)$, outputs the number of prime numbers less than or equal to x.
   (a) Calculate the following.
      i. $\pi(30)$
      ii. $\pi(31)$
      iii. $\pi(32)$
   (b) Is it true that $\pi(17) = \pi(18)$?
   (c) Is it true for every prime number p that $\pi(p) = \pi(p + 1)$?
4. The Fundamental Theorem of Arithmetic states that we can write any positive integer as a unique product of its prime factors.

   For example $24 = 2 \times 2 \times 2 \times 3$, $50 = 2 \times 5 \times 5$. This product is also known as the **prime-factorization** of an integer.
   (a) Determine the prime-factorization of the following integers.

i. 25
ii. 36
iii. 39
iv. 40

(b) We define $\Omega(n)$ to be the **Prime Omega Function** (since it's uppercase Omega). The function $\Omega(n)$ counts the number of prime factors of a positive integer n.

For example, $\Omega(24) = 4$ and $\Omega(50) = 3$.

For each integer in part (a), determine $\Omega(n)$.

(c) We define $\omega(n)$ to be the **prime omega function** (since it's lowercase omega). The function $\omega(n)$ counts the *unique* prime factors of a positive integer n. For example, $\omega(24) = 2$ since the unique prime factors are 2 and 3, we only count 2 once. The value of $\omega(50) = 2$ since we only count one of the 5's.

For each integer in part (a), determine $\omega(n)$.

(d) Consider the function given by $f(n) = \Omega(n) - \omega(n)$.

i. What can we say about n if $f(n) = 0$?
ii. What can we say about n if $f(n) \neq 0$?

5. Rewrite each sum as an expression that uses Sigma Summation Notation. As a hint, each of the sums start at $n = 1$.

(a) $3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 + 30$

(b) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$

(c) $\frac{3}{2} + 2 + \frac{9}{4} + \frac{12}{5}$

(d) The sum of all positive even numbers.

(e) The sum of all positive odd numbers.

(f) $0 + 1 + 1 + 2 + 1 + 2 + 1 + 3 + 2 + 2$, (Hint: use a function from Question 4)

6. In this question, we will look at some shortcuts for certain sums. You may use a calculator if it helps.

(a) Evaluate the following sums.

i. $\sum_{n=1}^{6} 1$
ii. $\sum_{n=1}^{15} 1$
iii. $\sum_{n=1}^{75} 1$

(b) Predict the value of $\sum_{n=1}^{k} 1$, where k is any positive integer.

(c) Evaluate the following sums.

i. $\sum_{n=1}^{5} 4$ ii. $\sum_{n=1}^{9} 11$ iii. $\sum_{n=1}^{10} \pi$

(d) Predict the value of $\sum_{n=1}^{k} c$, where c is a constant and k is any positive integer.

(e) For any positive integer k, $\sum_{n=1}^{k} n = \frac{k(k+1)}{2}$. Use this formula to evaluate the following sums.

i. $\sum_{n=1}^{7} n$ ii. $\sum_{n=1}^{34} n$ iii. $\sum_{n=1}^{72} n$

7. On January 1st, 2022, Constantine did one pushup. The next day he did two pushups, and the day after he did three pushups.

(a) Continuing this pattern of doing one more pushup than the day before, how many pushups in total did Constantine do in the month of January?

(b) Assume that for each day of 2022, Constantine does his number of pushups according to the day of the month (i.e. on the first day of every month, he starts again at one pushup and increases the pushups by one each day until the end of the month). How many pushups in total will he do for 2022?

(c) Constantine's sister, Constantina, does twice as many pushups as Constantine on any given day. How many pushups did she do in January?