Grade 9/10 Math Circles
March 9, 2022
Analytic Geometry

Introduction

We begin, as all math lessons lessons must, with a terrible joke.

A Mathematician's Joke*

A horse walks into a cafe, and consumes copious amounts of coffee. The barista says, "I'm going to have to cut you off, you're getting too jittery." The horse replies, "I think not!" and vanishes in a puff of smoke.

You see, this is a play on René Descartes' famous line, 'I think, therefore I am.' But, if I had told you that ahead of the joke, I'd be putting Descartes before the horse.

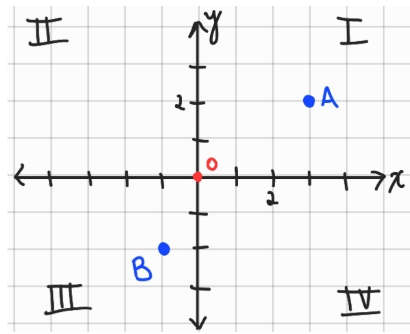
*I assume no responsibility for whether or not you find this funny.

René Descartes is not just the punchline of one of the worst jokes you've ever heard, but is widely considered the father of analytic geometry, which is also known as coordinate geometry. Analytic geometry ties together the Cartesian (I'm sure you can guess who that was named after!) coordinate system, and thereby algebra, to geometry. We won't be using any angles, similar/congruent triangle properties, or trigonometry here: we'll be building off of just our knowledge of points and lines, with a helping hand from the Pythagorean Theorem.

Cartesian Coordinates

Two Dimensions

You are likely quite familiar at this stage of your mathematical journey with the Cartesian coordinate system in two dimensions: this is just the xy-plane, as depicted on the next page. We have an x-axis (horizontal) and a y-axis (vertical), and we identify points on the plane by their (x, y) coordinates. The axes split the plane up into four quadrants, numbered as on the diagram. The location where the two axes meet, at $(0, 0)$, we call the origin (O). We tend to label points with capital letters.



Example 1

(a) What are the coordinates of A on the diagram above?

(b) What are the coordinates of the point that results from a reflection of A across the x-axis?

(c) What are the coordinates of the point that results from a reflection of A across the y-axis?

Solution:

(a) A is located at 3 along the x-axis and 2 along the y-axis, so it has coordinates $(3, 2)$.

(b) A reflection across the x-axis has no effect on the x-coordinate, and flips the sign of the y-coordinate, resulting in coordinates of $(3, -2)$.

(c) A reflection across the y-axis has no effect on the y-coordinate, and flips the sign of the x-coordinate, resulting in coordinates of $(-3, 2)$.

Exercise 1

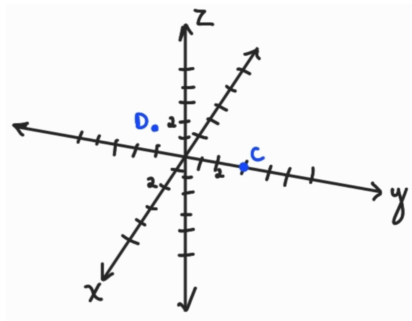
(a) What are the coordinates of B on the diagram above?

(b) What are the coordinates of the point that results from a reflection of B across the x-axis and then the y-axis?

Three Dimensions

The world that you and I live in is not two dimensional. Thankfully, we can extend the Cartesian coordinate system to three dimensions, as depicted on the next page. A good way to envision this is to stare at a floor/wall corner of the room you're sitting in right now and imagine it is the origin. Running along the floor/wall towards you is the x-axis, running along the other floor/wall is the

y-axis, and running up where the two walls meet is the newly introduced z-axis. This third axis brings in the new third dimension. We can identify the location of any point in three dimensional space via its (x, y, z) coordinate. The three axes now split up our space into eight octants, and they all meet at the origin, $(0, 0, 0)$.



Example 2

Which of C or D has coordinates $(1, -1, 2)$? Which has coordinates $(0, 3, 0)$?

Solution:

We can see that D is 1 along the x-axis, -1 along the y-axis, and 2 along the z-axis. We can see that C is 0 along the x-axis, 3 along the y-axis, and 0 along the z-axis.

Exercise 2

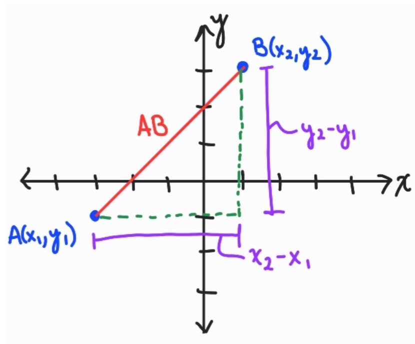
What are the three dimensional coordinates of the point that corresponds with the location of $(-2, 3)$ in two dimensional space? Draw this point.

Distance Between Points

One of the key tools in an analytic geometry toolkit is the ability to find the distance between two points. This is a place where the Pythagorean Theorem will come in handy.

Two Dimensions

Consider the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ as shown on the diagram on the next page. Connect these two points to form the line segment AB, whose length we want to know. If we draw a horizontal line through A and a vertical line through B we see that we form a right-angled triangle. Our goal is to find the length of the hypotenuse.



The length of the base of this triangle is the difference between the x-values of the two points, $x_2 - x_1$, and the height of this triangle is the difference between the y-values of the two points, $y_2 - y_1$. Then, application of the Pythagorean Theorem gives us our key result:

Distance Between Two Points in Two Dimensions

$$d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 3

Find the distance between $(-3, 4)$ and $(1, 0)$.

Solution:

Substituting into the formula, we find $d = \sqrt{(-3-1)^2 + (4-0)^2} = 4\sqrt{2}$.

Exercise 3

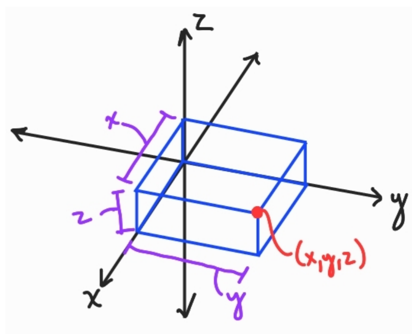
The points $(1, 2)$ and $(5, p)$ are a distance of 5 away from each other. Determine the possible values of p.

Exercise 4

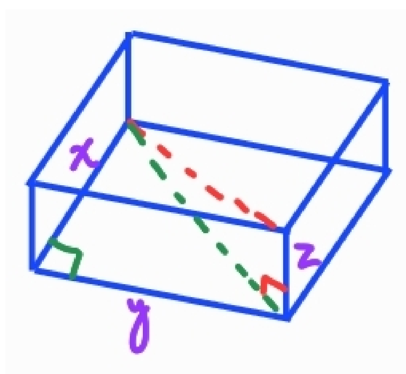
Determine the centre of a circle which passes through points $P(0, 4)$, $Q(2, 0)$, and $R(9, 1)$.
Hint: The radius of a circle is constant.

Three Dimensions

For the sake of ease, we will run this derivation by finding the distance between a point (x, y, z) and the origin, and then generalize this result to the distance between any two points in three dimensions. First, it is nice to visualize a point (x, y, z) as a vertex opposite the origin on a rectangular prism, as shown in the diagram below. The prism has length x, depth y, and height z.



Our goal is to find the distance of the diagonal between the origin and our desired vertex, which runs through the inside of the prism. This diagonal is in fact the hypotenuse of a right-angled triangle: one with a height of z and a base that runs the length of the diagonal of the base of the prism (itself a hypotenuse of another right-angled triangle). This is pictured below.



Using Pythagorean Theorem, the length of the base's diagonal is $\sqrt{x^2 + y^2}$. Then, plugging this into Pythagorean Theorem for the internal diagonal, we find that the distance between the origin and our desired vertex is

$$d = \sqrt{(\sqrt{x^2 + y^2})^2 + z^2}$$
$$d = \sqrt{x^2 + y^2 + z^2}$$
$$d = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2}$$

which we can generalize to the key result:

Distance Between Two Points in Three Dimensions

$$d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 4

Find the distance between $(1, -3, 4)$ and $(1, -1, 0)$.

Solution:

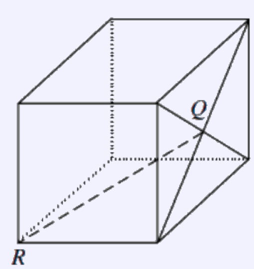
Substituting into the formula, we find $d = \sqrt{(1-1)^2 + (-3-(-1))^2 + (4-0)^2} = 2\sqrt{5}$.

Exercise 5

Determine the values of p for which the point $(1, 1, p)$ is the smallest possible integer distance away from the origin.

Exercise 6

Q is the point of intersection of the diagonals of one face of a cube of side length 2. What is the length of QR?



1998 Pascal Contest, Q21. Hint: Place R at the origin.

Midpoint of a Line Segment

In analytic geometry we also tend to concern ourselves with the midpoint of line segments connecting points of interest.

Two Dimensions

Consider the two points $A(x_1, y_1)$ and $B(x_2, y_2)$. The midpoint, $M(x_m, y_m)$, is the point the divides the line segment AB into two equal lengths. As such, the x and y-coordinates of M must be the average values of the x and y-coordinates of A and B, since M sits halfway along the x and y-distances between A and B. This gives rise to:

Midpoint of a Line Segment in Two Dimensions

$$M_{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 5

Verify that M is indeed the midpoint of AB by using the distance formula.

Solution:

Substituting into the formula, we find

$$\begin{aligned} d_{AM} &= \sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2} \\ &= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2} \\ &= \sqrt{\frac{1}{4}(x_2 - x_1)^2 + \frac{1}{4}(y_2 - y_1)^2} \\ &= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ d_{AM} &= \frac{1}{2} d_{AB} \end{aligned}$$

And since $d_{AB} = d_{AM} + d_{MB}$, we also find by substitution that $d_{MB} = \frac{1}{2} d_{AB}$. Thus, we have verified that M is indeed the midpoint of AB.

Exercise 7

Find the coordinates of the point that is $\frac{1}{4}$ of the distance from $A(-1, -1)$ to $B(3, 7)$.

Three Dimensions

We can quickly scale this concept to three dimensions:

Midpoint of a Line Segment in Three Dimensions

$$M_{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Exercise 8

Find the coordinates of the centre of a rectangular prism of height 2 along the z-axis, width 4 along the x-axis, and length 6 along the y-axis, residing in the positive regions of the x, y, and z axes, and whose corner is at the origin.

Lines

The other key component of analytic geometry is our knowledge of lines. We will only focus on two dimensions for the remainder of the lesson. Our key results for lines are:

Key Results for Lines

Slope-Intercept Form of a Line:

$y = mx + b$ where m is the slope, given by calculating $\dfrac{y_2 - y_1}{x_2 - x_1}$ for any two points along the line, and where b is the y-intercept.

Point-Slope Form of a Line:

$y = m(x - x_0) + y_0$, where m is the slope as above, and (x_0, y_0) is any point on the line.

Horizontal Lines:

A line of form $y = c$, where c is a constant.

Vertical Lines:

A line of form $x = c$, where c is a constant.

Parallel Lines:

Two lines whose slope, m, is the same.

Perpendicular Lines:

Two lines whose slopes, m_1 and m_2, are negative reciprocals. This means that $m_1 = -\frac{1}{m_2}$ or equivalently that $m_1 m_2 = -1$.

Example 6

Find the values of m_1 and b_1 such that $y_1 = m_1x + b_1$ is perpendicular to $y = 3(x - 1) + 2$ and shares its y-intercept.

Solution:

Expanding $y = 3(x - 1) + 2$ gives us $y = 3x - 1$. This equation has slope 3 and y-intercept -1. Thus, we will want $m_1 = -\frac{1}{3}$ and $b_1 = -1$.

Exercise 9

$A(-3, 2)$, $B(7, 2)$, and $C(-1, p)$ form a right-angled triangle with the right angle at point C. Determine all possible values of p, without using distance formulae or Pythagorean Theorem.

We can also find the intersection point of two lines:

Finding the Intersection Point of Two Lines

1. Ensure both equations are in $y =$ form.
2. Set both equations equal to each other, and solve for x.
3. Use x in either equation to solve for y.
4. The point of intersection is (x, y).

Example 7

Determine the point of intersection of $y = 3x + 4$ and $y = -2x - 1$.

Solution:

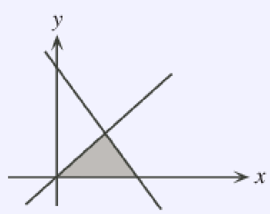
Setting the equations equal to each other and solving yields

$$\begin{aligned} 3x + 4 &= -2x - 1 \\ 5x &= -5 \\ x &= -1 \end{aligned}$$

And upon substitution into the first ($y = 3(-1) + 4$) or second ($y = -2(-1) - 1$) equations yields $y = 1$. Thus, the point of intersection is $(-1, 1)$.

Exercise 10

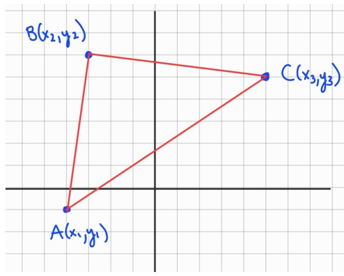
In the diagram, the shaded region is bounded by the x-axis and the lines $y = x$, and $y = -2x + 3$. What is the area of the shaded region?

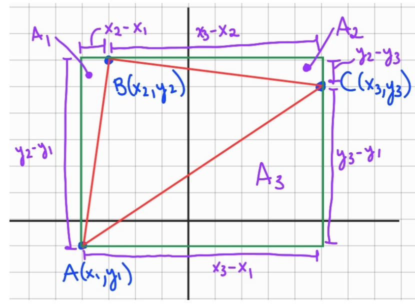


2008 Fermat Contest, Question 10

Triangles Bound by Coordinates

We conduct one final interesting derivation, for a general triangle bound by points $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$, as depicted on the next page. We will seek to find this triangle's area, by first making an ingenious choice: we will draw a bounding rectangle as shown in the second diagram.





Now all we have to do is take the rectangle's area and subtract off the three extraneous smaller right-angled triangles this construction has created. We have that the width of the rectangle is $x_3 - x_1$ and the height is $y_2 - y_1$. So far we then have:

$$A_{rect} = (x_3 - x_1)(y_2 - y_1)$$

Next, we focus on A_1. This triangle has a height of $y_2 - y_1$ and a base of $x_2 - x_1$. This gives us:

$$A_1 = \frac{1}{2}(x_2 - x_1)(y_2 - y_1)$$

Now, we focus on A_2. This triangle has a height of $y_2 - y_3$ and a base of $x_3 - x_2$. This gives us:

$$A_2 = \frac{1}{2}(x_3 - x_2)(y_2 - y_3)$$

Lastly, we focus on A_3. This triangle has a height of $y_3 - y_1$ and a base of $x_3 - x_1$. This gives us:

$$A_3 = \frac{1}{2}(x_3 - x_1)(y_3 - y_1)$$

Putting this all together, and working through the algebra gives

$$\begin{aligned}
A_{ABC} =& A_{rect} - A_1 - A_2 - A_3 \\
=& (x_3 - x_1)(y_2 - y_1) - \frac{1}{2}(x_2 - x_1)(y_2 - y_1) - \frac{1}{2}(x_3 - x_2)(y_2 - y_3) - \frac{1}{2}(x_3 - x_1)(y_3 - y_1) \\
=& \frac{1}{2}[2(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_2 - y_1) - (x_3 - x_2)(y_2 - y_3) - (x_3 - x_1)(y_3 - y_1)] \\
=& \frac{1}{2}[2x_3y_2 - 2x_3y_1 - 2x_1y_2 + 2x_1y_1 - x_2y_2 + x_2y_1 + x_1y_2 - x_1y_1 \\
& - x_3y_2 + x_3y_3 + x_2y_2 - x_2y_3 - x_3y_3 + x_3y_1 + x_1y_3 - x_1y_1] \\
=& \frac{1}{2}[x_3y_2 - x_3y_1 - x_1y_2 + x_2y_1 - x_2y_3 + x_1y_3]
\end{aligned}$$

Now, depending on which order you choose for your points, the quantity in square brackets may be negative, which is not permissible for area. Thus, we modify the formula using absolute values to arrive at:

Area of $\triangle ABC$ Bound by Coordinates $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$

$$A_{\triangle ABC} = \frac{1}{2}|x_3y_2 - x_3y_1 - x_1y_2 + x_2y_1 - x_2y_3 + x_1y_3|$$

Example 8

Find the area of the triangle bound by $(3, 3)$, $(-1, -2)$, and $(0, -1)$.

Solution:

Plugging into the formula yields

$$\begin{aligned} A &= \frac{1}{2}|x_3y_2 - x_3y_1 - x_1y_2 + x_2y_1 - x_2y_3 + x_1y_3| \\ &= \frac{1}{2}|(0)(-2) - (0)(3) - (3)(-2) + (-1)(3) - (-1)(-1) + (3)(-1)| \\ &= \frac{1}{2}|-1| \\ A &= \frac{1}{2} \end{aligned}$$

Exercise 11

If $A(2, 5)$, $B(-3, -6)$, and $C(p, 0)$ form a triangle of area 10, determine the possible values of p.

Hint: Note that $|x| = a$ leads to cases $x = a$ or $x = -a$.