Grade 9/10 Math Circles
March 30, 2022
Knot Theory

Introduction

This week, we will explore something different: the mathematical theory of *knots*. Here is how one constructs the kinds of knots we'll consider: begin with a piece of string, tie a knot in it, and then glue the ends of the string back together to form a closed loop (like the examples below).

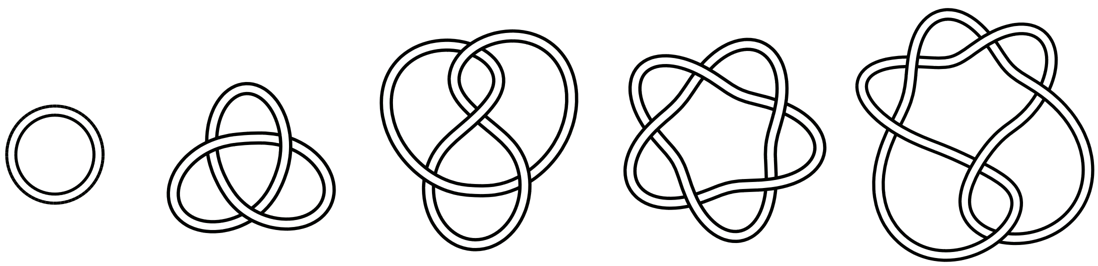

Figure 1: Some examples of knots[1]. From left to right these are called the *unknot*, the *trefoil* knot, the *figure-eight* knot, the *cinquefoil* knot, and *Stevedore's* knot. The nicest possible knot is the unknot: it has no knotting at all!

It seems intuitively obvious that we can't undo the knots on the right without cutting the string somewhere, but how can we prove this? Similarly, it seems like these knots are all different, but how can we tell? In this lesson, we will look at a few ways to represent and distinguish knots, along with some of the history of knot theory.

A little bit of history

In fact, knot theory (as a field of mathematics) is over 100 years old! In the 1880's, in an effort to explain the physics of light, scientists hypothesized that matter was permeated by a universal substance called the *ether*. Taking this one step further, the chemist Lord Kelvin William Thomson proposed that the different chemical elements were simply "knots" in this ether.

While more accurate models of electromagnetic waves proved this theory to be false, mathematicians and physicists were already hard at work trying to understand knots for their own sake, and the

[1] Many of the images in this worksheet were created using *KnotPlot*. You can find a link to this software (and other resources) at the end of this document.

mathematical theory of knots was born. More recently, biologists and chemists have discovered that molecules (like DNA) can be knotted, and so knot theory now has significant physical applications.

An introduction to knot projections

First, let's give a quick definition of a knot.

Definition:
A **knot** is a closed loop in 3-dimensional space.

We will consider two knots to be the same if we can get from one to the other by moving them around in 3-dimensional space. In other words, we can tangle (or un-tangle) a knot as much as we like, but we can't cut the knot or pass it through itself.

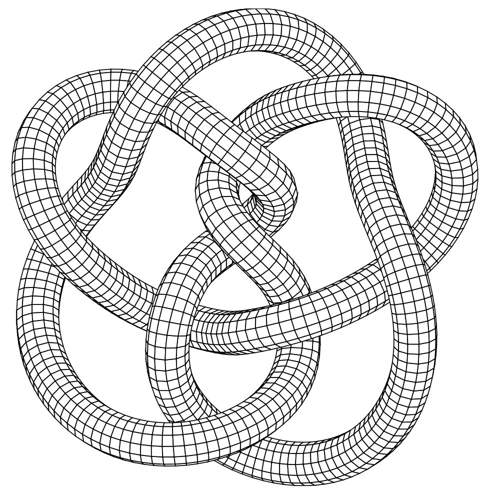

Figure 2: A knot drawn as a loop in 3D.

Instead of trying to draw pictures of knots in 3D, we will draw 2D schematics of knots called *knot diagrams* like the one below. Simply imagine laying the knot in Figure 2 flat on the surface of a table to get the picture in Figure 3.

At certain points in the diagram, the knot passes over or under itself. These are called **crossings** of the diagram. For example, the diagram in Figure 3 has 9 crossings.

Exercise 1:
The trefoil knot in Figure 1 has three crossings. How many crossings do the other knots in Figure 1 have?

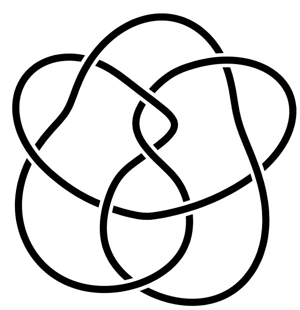

Figure 3: A 2D knot diagram of the knot from Figure 2.

The **crossing number** of a knot K, denoted $c(K)$, is the least number of crossings needed in any diagram for K. Most knot tables (like the one at the end of this lesson) are organized by crossing number. For example, the trefoil has crossing number equal to 3, since it has a diagram with 3 crossings, but does *not* have a diagram with only 2 crossings.

We can produce tables of knots organized by their crossing number: a bit like prime numbers or a periodic table or elements. Some of the first knot tables (up to 10 crossings) were produced by Peter

Guithrie Tait and Charles Newton Little in the late 1800's. On the last page of this document, you will find a knot table with all of the distinct knots with crossing number at most 7. The knots are named according to their position in this table; for example, the trefoil is called 3_1 because it is the first knot in the table with crossing number equal to 3.

These days, modern supercomputers can tabulate all knots with up to about 20 crossings. There are 352,152,252 knots[2] with crossing number ≤ 20.

Example 1:

Any knot diagram with only one crossing must be a diagram for the unknot! If a knot diagram has only one crossing, then the ends can only be joined up in (essentially) two different ways! The two possibilities are illustrated below, and it is easy to see that they can both be unknotted.

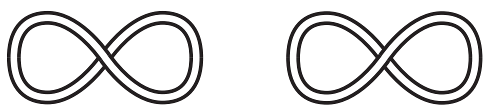

Figure 4: The only two diagrams with one crossing (up to moving the knot around a bit), both of which are unknotted.

The same fact is also true for diagrams with only two crossings.

Exercise 2:

Prove that any knot diagram with only two crossings is a diagram for the unknot.

In particular, this means that the simplest non trivial knot must the trefoil (Figure 1), which has three crossings. If we can prove that this is actually a non-trivial knot (which we will do later), then we know that it cannot have a diagram with less than three crossings.

In general, the number of knots with a given crossing number grows very quickly! The OEIS sequence which records the number of knots with crossing number n is:

$$0, 0, 1, 1, 2, 3, 7, 21, 49, 165, 552, 2176, 9988, 46972, 253293, 1388705, 8053393, 48266466, \ldots$$

In particular, the trefoil knot is the unique knot with crossing number equal to three, and the figure eight knot is the unique knot with crossing number equal to four.

[2] This is a tabulation of *prime* knots, which are the analogue of prime numbers for knots.

Exercise 3:

Find diagrams for the knot 3_1 with 4, 5, and 6 crossings. Can you find a diagram for 3_1 with 100 crossings? Why does this not contradict the fact that $c(3_1) = 3$?

Be careful: even if we have a diagram of a knot K with n crossings, this doesn't mean that $c(K) = n$, since it's possible that there is a completely different diagram with fewer crossings. The diagrams in the table at the end of this lesson have the *least* possible number of crossings.

The unknotting number

One important question in knot theory is the following: how can we recognize whether a knot diagram describes the unknot? Maybe you're able to untangle a knot easily– but what if you can't? How long should you continue to try before you decide it's not the unknot?

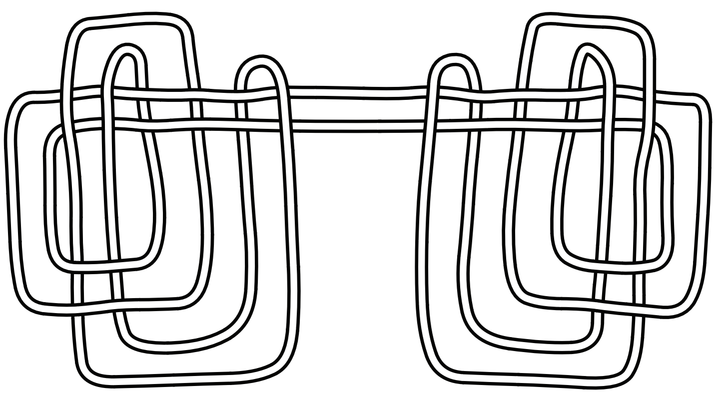

Figure 5: This diagram is known as *Freedman's unknot*, after the mathematician who produced it. In a precise sense, this diagram is "hard" to untangle (especially for computers).

Exercise 4:

The knot diagram in Figure 5 is actually unknotted! However, this is a surprisingly "hard" diagram. Can you show that this knot diagram is indeed the unknot?

Hint: Instead of trying to visualize this process, you can work with the diagram like the example below. If you have a chalkboard or a dry-erase board, this makes things even easier!

In a precise sense, checking whether a diagram describes the unknot is a very hard mathematical

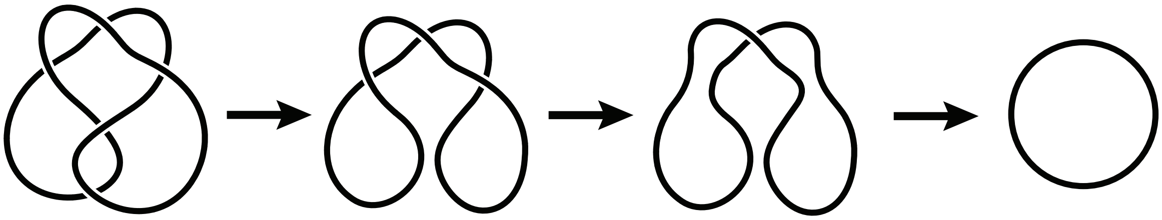

Figure 6: Modifying diagrams in steps to check that a certain diagram is the unknot.

problem. Similarly, deciding whether two knots are the same is also very hard!

Example 2:

There is a famous example of two knots that were long thought to be different, but turned out to be the same! The *Perko pair* is a supposed pair of distinct knots in Little's knot table from the late 1800's that eventually found its way into modern books of knot tables. In 1973, while trying to find different ways of distinguishing knots, Kenneth Perko discovered that these two knots are actually the same.

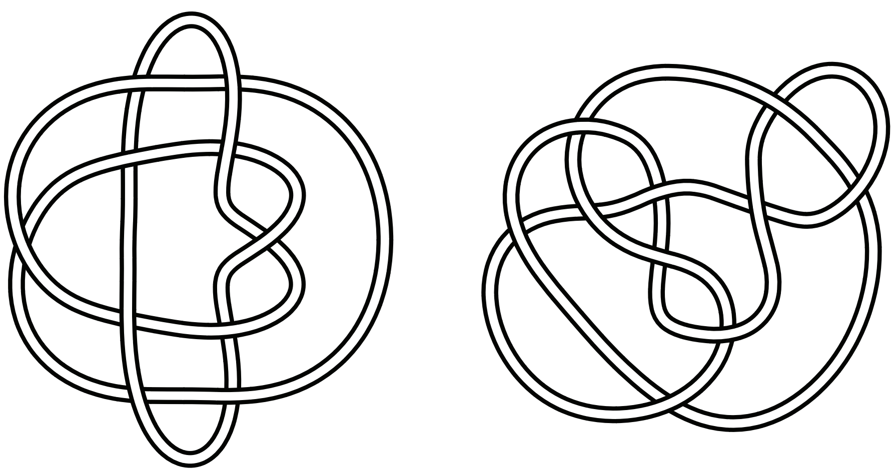

Figure 7: A pair of 10 crossing knots that were thought to be different for most of the 1900s!

Exercise 5:

Show that the Perko "pair" of knots are really just the same knot.

Next, we'll discuss another number that we can associate to a knot, called the unknotting number. Suppose that we have a diagram for a knot K, which has some number of crossings.

Definition:

A **crossing change** is the result of changing an "over" crossing to an "under" crossing, or vice versa.

Figure 8: A *crossing change* switches one of the crossings in a knot diagram.

The **unknotting number** of a knot K, denoted $u(K)$, is the *least* number of crossing changes we can perform to produce the unknot.

Exercise 6:

For each of the diagrams in Figure 1 and Figure 7, find some crossing changes so that the resulting diagram is unknotted.

Be careful! If you can find a sequence of m crossing changes for K that produces the unknot, this doesn't mean that $u(K) = m$; it only means that $u(K) \leq m$. In general, there could be a completely different sequence of crossing changes (or even a different diagram) with fewer steps.

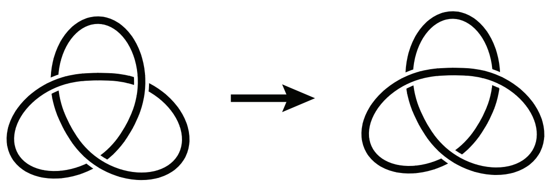

Figure 9: Changing one of the crossings of the trefoil produces the unknot.

It may not seem obvious, but if we start with a knot diagram, we can *always* change some of the crossings so that the resulting diagram is unknotted. For example, if we start with the trefoil knot, changing any one of the crossings produces the unknot.

Exercise 7:

Show that given any diagram for a knot, there is always a way to change some of the crossings so that the resulting diagram is unknotted (try the knots in the table first).

Hint: Pick a starting point and move around the knot in some direction. As you move around, change crossings so that you always cross *over* any part of the knot you have already visited.

This last exercise implies that that for any knot K, we have the inequality

$$u(K) \leq c(K).$$

Indeed, if we have a diagram for K with exactly $c(K)$ crossings, then this exercise shows that there is a way to change at most this many crossings to produce the unknot, and so $u(K) \leq c(K)$. In fact, this argument can be improved to give the inequality $u(K) \leq \frac{1}{2}c(K)$.

Reidemeister moves

Now that you have experience working with knot diagrams, you may have observed that all changes to a knot diagram can be broken down into the following three basic moves:

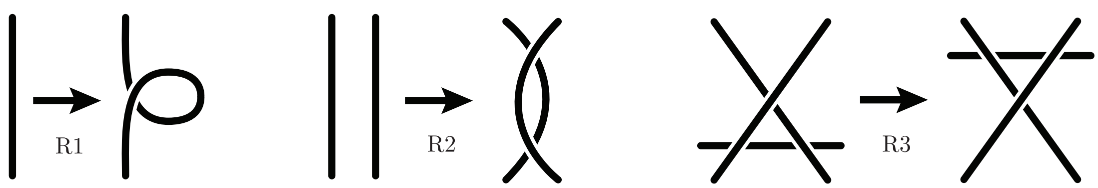

Figure 10: The three *Reidemeister moves* R1, R2, and R3. These take place in a small part of a knot diagram.

Stop and Think

If we change a diagram by one of these moves, do we change the knot it describes? Why or why not?

It is a theorem (that we won't prove) that any two diagrams for the same knot can be related by a sequence of these basic moves. They are called **Reidemeister moves**, after one of the mathematicians who proved this theorem.

Theorem: (Reidemeister, 1927)
Any two knot diagrams for the same knot are related by a sequence of Reidemeister moves.

If we pick two diagrams that describe the same knot, it is a very hard problem to decide how many moves we need to go from one diagram to the other. However, these moves are still extremely useful!

Exercise 8:
Use Reidemeister moves to show that the two diagrams in Figure 11 describe the same knot.

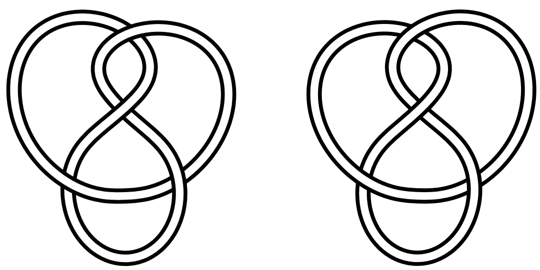

Figure 11: On the left, the figure eight knot, and on the right, the *mirror* of the figure eight knot. Remarkably, these are the same!

One way to tell knots apart

So far, we have been able to show that some knot diagrams describe the same knot– by finding a sequence of Reidemeister moves that takes one diagram to the other. However, what if we start with knot diagrams for *different* knots? How could we distinguish them?

One way to do this is with **knot invariants**. An invariant is simply anything we can assign to a knot that doesn't depend on a particular diagram.

Example 3:
We've already seen two knot invariants! For example, the crossing number $c(K)$ and unknotting number $u(K)$ are both numbers we assign to a knot K. Even though they can be very hard to compute, they don't depend on a diagram for K.

Now, we'll look at a knot invariant called **tricolorability**, which is much easier to compute than either $c(K)$ or $u(K)$.

Definition:

A knot diagram is called **tricolorable** if the arcs in the diagram can be colored red, blue, and green so that:

- At least two colors are used;
- Each crossing is colored by either all three colors, or only one color.

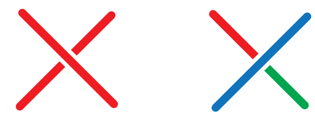

Figure 12: The two possibilities at a crossing when we tricolor a knot; all one color (e.g., red) or all different colors.

For example, the trefoil knot *can* be tricolored! However, there are diagrams that cannot be tricolored.

Exercise 9:

Show that the diagrams for the figure eight knot and for the unknot in Figure 1 can't be tricolored.

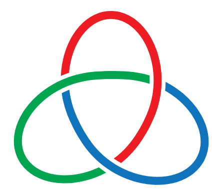

Figure 13: A tricoloring of the trefoil knot.

Whether a knot is tricolorable looks like a good invariant– for instance, it seems to distinguish the trefoil knot from the unknot! However, we've only shown that one particular diagram for the trefoil knot is tricolorable. Is it true that *all* knot diagrams for the trefoil are tricolorable? Similarly, we have one diagram for the unknot which is not tricolorable; is it the case that *all* knot diagrams for the unknot fail to be tricolorable? Fortunately, the following theorem guarantees that the property of tricolorability really is a knot invariant!

Try to read through the proof carefully (or you can simply take this fact for granted).

Theorem:

All diagrams for a knot K are tricolorable, or none of them are.

To prove this theorem, we will use Reidemeister moves! Remember: any two diagrams for K are related by a sequence of these elementary moves (they are illustrated in Figure 10).

We will show that if we start with a diagram which is tricolorable, then the result of doing any Reidemeister move is still a tricolorable diagram. Since we can get between any two diagrams by

Reidemeister moves, this means that all diagrams for K are tricolorable, or none of them are. To do this, we will examine one Reidemeister move at a time.

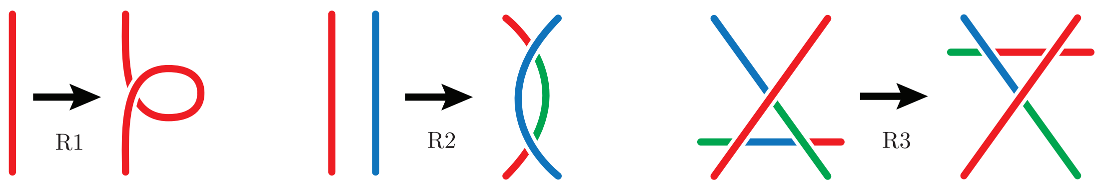

Figure 14: Performing any of the three Reidemeister moves preserves the tricolorability property.

The R1 move:

Suppose that we start with a tricolorable diagram, and then perform an R1 move. This move only involves one strand; we will assume it is colored red. After doing the R1 move, we see that we can still tricolor the new diagram![3]

The R2 move:

Again, we will suppose that we start with a tricolorable diagram. This time, we will perform an R2 move. This move involves two strands; we will assume that they are colored red and blue. After performing the R2 move, we can still tricolor the diagram (you should check the other possible colorings for the original strands).

The R3 move:

Once again, we will suppose that we start with a tricolorable diagram, and then perform an R3 move. This move involes *three* strands, so there are a few possibilities for colors (the illustration shows one example). For any possible coloring of these strands, the resulting diagram can still be tricolored (you also should check the remaining possibilities for this case).

Exercise 10:

What other knots in the knot table on page 11 are tricolorable?

This fact guarantees that tricolorability is a well defined knot invariant. Since we know that the trefoil is tricolorable but the unknot is not, this gives us a (mathematical) proof that the trefoil knot cannot be unknotted! You can find many other interesting invariants for knots on the website *KnotInfo* listed on the next page.

[3]A small note: the original coloring must have used all three colors, so the new one does too.

A small knot table and other cool resources

Here are some online knot theory resources:

- The Knot Atlas: an online atlas of knots, with interesting facts, pictures, and more
- KnotPlot: free software for viewing knots in 3D
- KnotInfo: an online resource where you can find many more invariants of knots