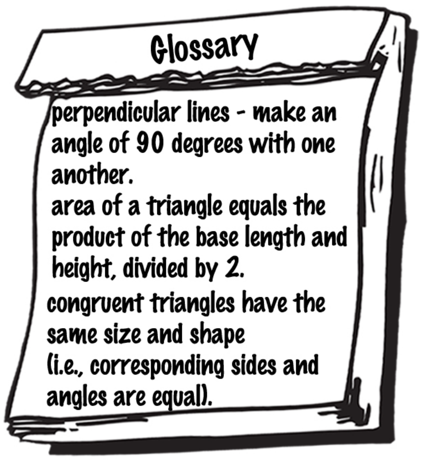
Problem

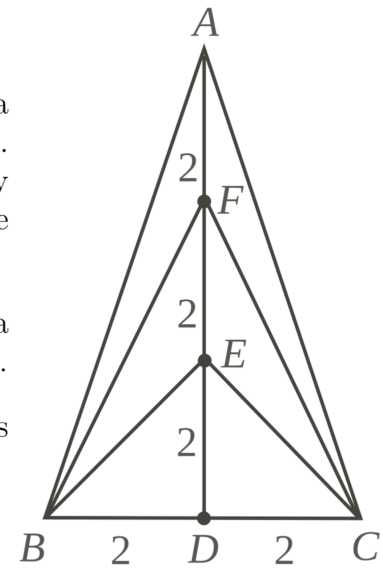
In the triangle ABC, AD is perpendicular to BC, and each of the line segments AF, FE, ED, BD, and DC has a length of 2 units.



a) Colour a section of the figure with area equal to 1/2 the area of triangle ABC. Discover how to do this in as many different ways as you can, using the smaller triangles.

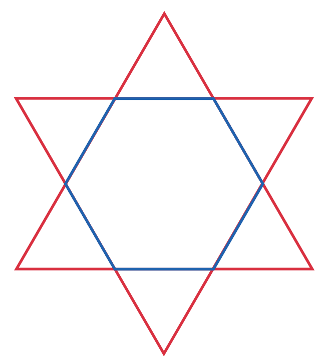
b) Repeat part a) for regions with area equal to 1/3 the area of triangle ABC.

c) How many pairs of congruent triangles are there in this diagram?



Extension **:**

1. In the diagram at right, each line segment has the same length.
   a) What type of triangles are on the exterior (outside) of the figure?
   b) How many of these triangles are needed to fill the interior hexagon?



Hints

Hint 1 - a) What is the area of triangle ABC?

Hint 2 - a) What is the height of triangle BDE? Triangle BEF? Triangle BAF?

Hint 3 - a) What is the area of triangle BAF?

Hint 4 - c) Are triangles of equal area necessarily congruent?

Extension:

Hint 1 - What figures appear when you draw the three diagonals through the centre of the hexagon?

Suggestions:

1. You may wish to review the area formula for triangles, and the meaning of 'congruent' beforehand.

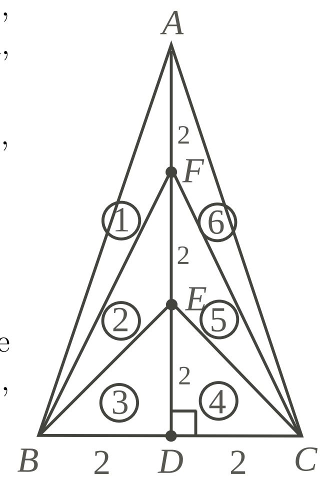
2. For many students, the Extension can be solved easily if the 'star' is cut out very carefully, and the outer triangles cut off, or folded along the base of each triangle.

3. This problem could be done in small groups.

Solution

The KEY IDEA here is that each of the six smallest triangles in the diagram has the same height (2) and base length (2). Note that this uses the fact that for an oblique triangle such as ABF (or FBE), the 'height' is the length of the perpendicular BD to the extended base AD (or FD).

Hence all six of these triangles have the same area, equal to $\frac{1}{6}$ of the area of triangle ABC.



a) To colour a section of the figure with area equal to $\frac{1}{2}$ that of the triangle ABC, all that is needed is to colour any three of the six smallest triangles, since $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2}$. Using the numbered triangles in the diagram, the 20 different ways to do this are:

(1,2,3), (1,2,4), (1,2,5), (1,2,6), (1,3,4), (1,3,5), (1,3,6), (1,4,5), (1,4,6), (1,5,6);

(2,3,4), (2,3,5), (2,3,6), (2,4,5), (2,4,6), (2,5,6);

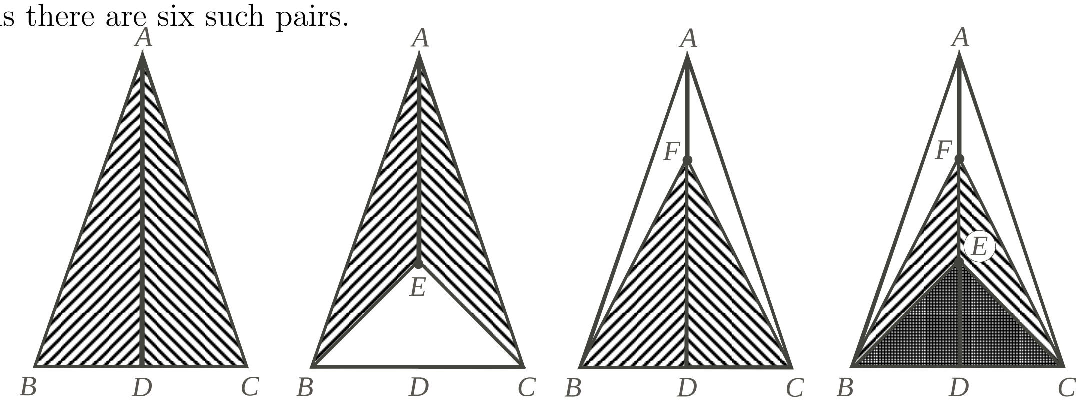
(3,4,5), (3,4,6), (3,5,6); (4,5,6).

b) To colour a section of the figure with area equal to $\frac{1}{3}$ that of the triangle ABC, all that is needed is to colour any two of the six smallest triangles, since $\frac{1}{6}+\frac{1}{6}=\frac{2}{6}=\frac{1}{3}$. There are 15 different ways to do this:

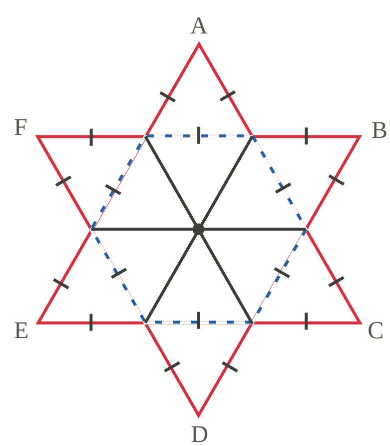
(1,2), (1,3), (1,4), (1,5), (1,6); (2,3), (2,4), (2,5), (2,6);

(3,4), (3,5), (3,6); (4,5), (4,6); (5,6).

c) Using the symmetry of the figure about AD, the pairs of congruent triangles are: ABD and ADC, ABE and AEC, FBD and FDC, ABF and AFC, FBE and FEC, and EBD and EDC. Thus there are six such pairs.



Extension:



1.a) Since each line segment has the same length, the triangles on the outside of the figure have all three sides of equal length, and hence are equilateral triangles, with each angle being 60°.

b) If the figure is cut out, and the six exterior triangles are folded along the dotted lines as shown in the diagram, the vertices A, B, C, D, E, F all meet at the centre of the hexagon, with the lateral sides matching. Thus the six exterior triangles exactly fill the interior of the hexagon. (See Emmy Noether Circle 1 for 2010-2011, Problem 6 d) for a complementary problem about hexagons.)