Problem of the Month
Solution to Problem 0: September 2019

Solution

Solution 1

We will assume that P, Q, and R are lattice points and that $\angle PQR = 60^\circ$. Our goal is to show that these assumptions lead to a contradiction.

First of all, we make three simplifying claims.

Claim 1: We can assume that Q is at the origin.

Proof of Claim 1: Suppose Q has coordinates (u, v). Translating P, Q, and R horizontally by $-u$ and vertically by $-v$ will preserve the fact that $\angle PQR = 60^\circ$. Since Q is a lattice point and u and v are integers, these translations move Q to the origin and preserve the fact that P and R are lattice points.

Claim 2: We can assume P is in the first quadrant.

Proof of Claim 2: If P and R are both reflected in the same axis, they will still be lattice points and the measure of $\angle PQR$ will still be 60° (we are assuming Q is at the origin). By reflecting in one or both axes, we can move P to the first quadrant without changing the measure of $\angle PQR$ or the fact that P and R are lattice points.

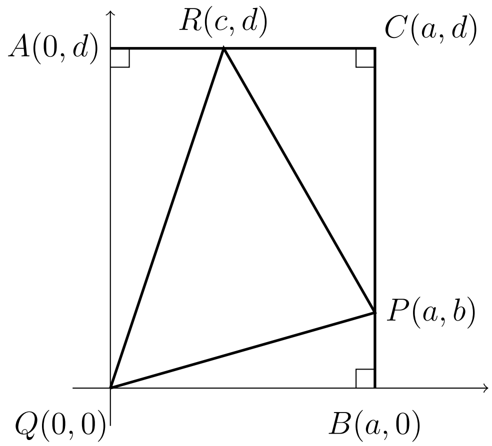
Claim 3: We can assume that ray QR makes a larger angle with the positive x-axis than ray QP does. We mean for these angles to be measured counterclockwise from the positive x-axis. That is, if R were the point $(1, -1)$, we would take the angle QR makes with the positive x-axis to be 315°, and not 45°.

Proof of Claim 3: Reflecting both P and R in the line $y = x$ will preserve $\angle PQR$ and the fact that P and R are lattice points. By possibly reflecting P and R in the line $y = x$, we can ensure that P and R satisfy the conditions in the claim.

Labelling $P(a, b)$ and $R(c, d)$, Claim 2 allows us to assume that $a \geq 0$ and $b \geq 0$. Together with the facts that $\angle PQR = 60^\circ$ and P is in the first quadrant, Claim 3 allows us to assume $d \geq 0$.

We will compute the area of $\triangle PQR$ in two different ways, equate the areas, then derive our contradiction.

Although our claims have simplified things, there are still several different ways that the points P and R could be configured relative to each other. For now, we assume $0 < c < a$ and that $b < d$. We will compute the area of $\triangle PQR$ in terms of a, b, c, and d. To help with the computation, label $A = (0, d)$, $B = (a, 0)$, and $C = (a, d)$ and consider the diagram below.



Because of how points A, B, and C are placed, we have that $\triangle QAR$, $\triangle RCP$, and $\triangle PBQ$ are right triangles. Furthermore, the area of $\triangle PQR$ can be computed by subtracting the areas of these right triangles from the area of rectangle $QACB$. If we let t denote the area of $\triangle PQR$, we then have

$$\begin{aligned} t &= (QB)(QA) - \frac{1}{2}(AR)(AQ) - \frac{1}{2}(BQ)(BP) - \frac{1}{2}(CP)(CR) \\ &= ad - \frac{1}{2}cd - \frac{1}{2}ab - \frac{1}{2}\big[(d-b)(a-c)\big] \\ &= ad - \frac{1}{2}cd - \frac{1}{2}ab - \frac{1}{2}(ad - cd - ab + bc) \\ &= \frac{1}{2}\big[2ad - cd - ab - ad + cd + ab - bc\big] \\ &= \frac{1}{2}(ad - bc) \end{aligned}$$

We have shown that when $0 < c < a$ and $b < d$, the area of $\triangle PQR$ is $\frac{1}{2}(ad-bc)$.

In all other similar situations, the diagram and calculations are slightly different, but the area will still be $\frac{1}{2}(ad-bc)$ as long as P, Q, and R satisfy Claims 1-3. For example, if $c=0$, then $\triangle PQR$ can be taken to have base RQ and height QB, so its area is $\frac{1}{2}(RQ)(QB) = \frac{1}{2}ad$. However, we are assuming $c = 0$, so the area is still $\frac{1}{2}(ad-0) = \frac{1}{2}(ad-bc)$. You might want to identify another case and check for yourself!

To compute the area of $\triangle PQR$ in a different way, we will use a reasonably well-known formula which says that the area of $\triangle PQR$ is equal to $\frac{1}{2}(QP)(QR)\sin\angle PQR$. You might want to try to derive this formula by dropping a perpendicular from R to the line containing PQ. The lengths QP and QR can be computed using the Pythagorean Theorem: $QP = \sqrt{a^2+b^2}$ and $QR = \sqrt{c^2+d^2}$. Here, the fact that c could be negative is irrelevant since we are squaring it. We are assuming that $\angle PQR = 60°$, so $\sin\angle PQR = \frac{\sqrt{3}}{2}$. Equating the two areas, we have

$$\frac{1}{2}(ad-bc) = \frac{1}{2}\frac{\sqrt{3}}{2}\sqrt{a^2+b^2}\sqrt{c^2+d^2}.$$

Multiplying both sides by 4, we get

$$2(ad-bc) = \sqrt{3}\sqrt{a^2+b^2}\sqrt{c^2+d^2},$$

and squaring both sides gives

$$4(ad - bc)^2 = 3(a^2 + b^2)(c^2 + d^2).$$

Expanding, this gives

$$4a^2d^2 - 8abcd + 4b^2c^2 = 3a^2c^2 + 3a^2d^2 + 3b^2c^2 + 3b^2d^2.$$

After rearranging, we get

$$a^2d^2 - 2abcd + b^2c^2 = 3a^2c^2 + 6abcd + 3b^2d^2$$

which can be factored to get

$$(ad)^2 - 2(ad)(bc) + (bc)^2 = 3[(ac)^2 + 2(ac)(bd) + (bd)^2],$$

and some more factoring leads to

$$(ad - bc)^2 = 3(ac + bd)^2.$$

Next, we will show that $ad - bc = 0$ by showing that if $ad - bc = 0$, then P, Q, and R lie on a line and so do not form a triangle. To see this, suppose $ad - bc = 0$ which means $ad = bc$. If $a \neq 0$ and $c \neq 0$, then $\frac{b}{a} = \frac{d}{c}$, so all of P, Q, and R lie on the line $y = mx$ where $m = \frac{b}{a}$ (recall that Q is at the origin). If $a = 0$, then $b \neq 0$. This is because P and Q are assumed to be different, so a and b cannot both be 0. However, $a = 0$ implies $ad = 0$, and since $ad = bc$, this means $bc = 0$. Thus, since $b \neq 0$ we have $c = 0$. In this case, P, Q, and R all lie on the y-axis. Using similar reasoning, if $c = 0$, then a must be 0, which means all three points are on the y-axis. Therefore, $ad - bc \neq 0$, as claimed.

We know that $(ad - bc)^2 = 3(ac + bd)^2$ and have just shown that the left side of this equation is not 0. This means we also have $ac + bd \neq 0$. Therefore, in the equation $(ad - bc)^2 = 3(ac + bd)^2$, we can divide by $(ac + bd)^2$ to get

$$\left(\frac{ad - bc}{ac + bd}\right)^2 = 3,$$

and after taking square roots, we get

$$\left|\frac{ad - bc}{ac + bd}\right| = \sqrt{3}.$$

Since a, b, c, and d are integers, this implies $\sqrt{3}$ is rational. It is well-known that $\sqrt{3}$ is irrational, which gives a contradiction. This can only mean our assumption that $\angle PQR = 60^\circ$ when P, Q, and R lattice points cannot have been true. In other words, if P, Q, and R are lattice points, then the measure of $\angle PQR$ cannot be 60°.

Note

If you are familiar with vectors, norms, and the dot product, you might be interested in an alternate way to start Solution 1. As in the beginning of the solution, we can translate Q to the origin. This would make the sides QP and QR of $\triangle PQR$ vectors with tails at the origin and heads at P and R. For any vectors $\vec{u}$ and $\vec{v}$ in the plane, there is a well-known formula which says $\vec{u} \cdot \vec{v} = \cos(\theta)\|\vec{u}\|\|\vec{v}\|$ where $\vec{u} \cdot \vec{v}$ is the dot product of $\vec{u}$ and $\vec{v}$, $\|\vec{u}\|$ and $\|\vec{v}\|$ are the norms of $\vec{u}$ and $\vec{v}$ respectively, and θ is the angle between $\vec{u}$ and $\vec{v}$. If P is the point (a, b) and R is the point (c, d) and we suppose the angle between QP and QR is $60°$, then this leads to

$$(a, b) \cdot (c, d) = \cos(\theta)\|(a, b)\|\|(c, d)\|$$

which is the same as

$$ac + bd = \frac{1}{2}\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}.$$

Multiplying both sides by 2 and squaring gives

$$4(ac + bd)^2 = (a^2 + b^2)(c^2 + d^2).$$

After expanding and rearranging, this would lead to

$$\left|\frac{ad - bc}{ac + bd}\right| = \sqrt{3}$$

as in the solution above.

<u>Solution 2</u>

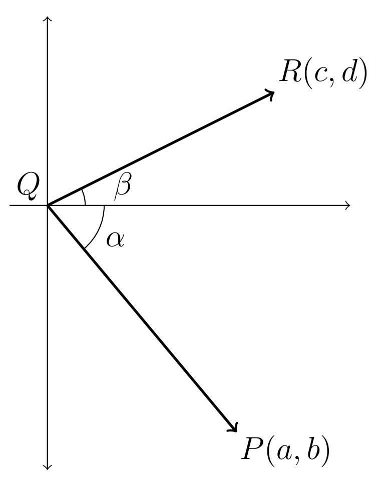
Assume that P, Q, and R are lattice points. We will first make a few simplifying assumptions about the points P, Q, and R. For additional justification of why these assumptions can be made safely, see the beginning of Solution 1.

By possibly translating all three points the same distance horizontally and vertically, we can ensure that Q is at the origin without changing the measure of $\angle PQR$ or the fact that P, Q, and R are lattice points.

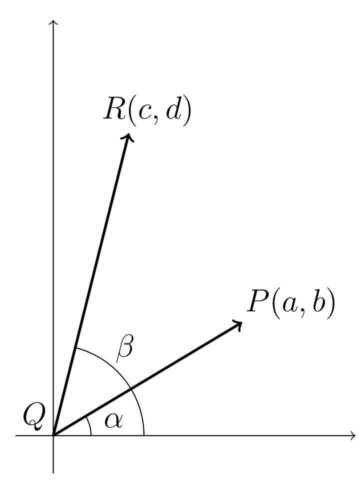
If the measure of $\angle PQR$ is greater than $90°$, then $\angle PQR \neq 60°$, so there is nothing to prove. Otherwise, P and R must either be in the same quadrant or in adjacent quadrants. By possibly reflecting in one or both of the axes, and possibly reflecting in the line $y = x$, we can ensure that P and R are lattice points somewhere in the part of the plane covered by the first and fourth quadrants. That is, we can assume that the x-coordinates of P and R are both positive. The point Q is at the origin, so none of these reflections will change $\angle PQR$.

To summarize, we are assuming Q is at the origin and that P and R are in the right half of the plane. We now let α be the smallest positive angle that ray QP makes with the positive x-axis, and β be the smallest positive angle that ray QR makes with the positive x-axis.

Suppose the coordinates of P are (a, b) and the coordinates of R are (c, d). Depending on how P and R are positioned, we have either $\angle PQR = \alpha + \beta$, $\angle PQR = \alpha - \beta$, or $\angle PQR = \beta - \alpha$. Two such situations are pictured below. There are several others.



$\angle PQR = \alpha + \beta$



$\angle PQR = \beta - \alpha$

If $a = 0$, then P is on the y-axis. Since R is in the first or fourth quadrant, if $\angle PQR$ were 60°, then we would have $\beta = 30^\circ$ in this situation, which would mean the slope of the line through Q and R is $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$. On the other hand, the slope of this line is $\dfrac{d}{c}$. Equating these two slopes and rearranging gives $\sqrt{3} = \dfrac{c}{d}$. It is well known that $\sqrt{3}$ is irrational, so this cannot happen if c and d are both integers. Therefore, if $a = 0$, then the measure of $\angle PQR$ cannot be 60°.

Similarly, if $c = 0$, then the measure of $\angle PQR$ cannot be 60°.

Otherwise, $\tan(\alpha)$ and $\tan(\beta)$ are both defined and equal to $\dfrac{b}{a}$ and $\dfrac{d}{c}$, respectively.

Suppose $\angle PQR = \alpha + \beta$. If $\angle PQR = 90^\circ$, then $\angle PQR \neq 60^\circ$, so there is nothing to show. Otherwise,

$$\tan \angle PQR = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{bd}{ac}}$$

which is rational because a, b, c, and d are integers. Since $\tan 60^\circ = \sqrt{3}$ is irrational, this means $\angle PQR$ is not 60°.

Suppose $\angle PQR = \alpha - \beta$. Similar to above, if $\angle PQR = 90^\circ$ then we are done. Otherwise,

$$\tan \angle PQR = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{\frac{b}{a} - \frac{d}{c}}{1 + \frac{bd}{ac}}.$$

This is a rational value when a, b, c, and d are integers. Therefore, since $\tan 60^\circ$ is irrational, it is not possible for the measure of $\angle PQR$ to be 60°.

Finally, a similar argument shows that if $\angle PQR = \beta - \alpha$, then $\tan \angle PQR$ is rational, and so $\angle PQR$ cannot measure 60°.

Therefore, if P, Q, and R are lattice points, then the measure of $\angle PQR$ cannot be 60°.