Problem of the Month

Solution to Problem 2: November 2019

Solution

(a) We first observe that if a sequence of moves is performed, the order in which the moves are performed does not affect the overall outcome. To prove this, it is enough to show that if two different moves are performed in either order, the same result is achieved.

Suppose Move A and Move B both involve the i^{th} dial. In particular, let us suppose Move A rotates the i^{th} dial by a positions and Move B rotates the i^{th} dial by b positions. Here, a and b are allowed to be integers and a negative integer represents a counterclockwise rotation. The net effect of Move A followed by Move B on the i^{th} dial is a rotation by $a + b$ positions. The net effect of Move B followed by Move A is a rotation by $b + a = a + b$ positions. Therefore, the net effect on the i^{th} dial is the same regardless of the order in which the moves are performed.

If the i^{th} dial is affected by Move A and not by Move B, then the net effect of Move A and Move B on the i^{th} dial is the same as the effect of Move A, regardless of the order. Similarly, if the i^{th} dial is affected by Move B and not by Move A, then the net effect is the same regardless of the order in which the moves are performed.

Of course, if neither Move A nor Move B has any effect on the i^{th} dial, then the effect of Move A and Move B on the i^{th} dial is the same regardless of the order in which the moves are performed.

It follows that the final state of the dials only depends on which moves were performed and not on the order in which they were performed. Also, there is no reason to rotate the same pair of dials twice since the same effect could be achieved with a single rotation. Furthermore, there is no reason to rotate the same pair of dials by more than three positions in either direction since this would result in the dial going all the way around at least once. The same effect could be achieved using a rotation by fewer positions. Finally, the effect of a rotation in the counterclockwise direction can be achieved by a rotation in the clockwise direction (possibly by a different number of positions). Therefore, there is no reason to rotate any pair of dials in the counterclockwise direction.

There are $k-1$ possible pairs of dials that can be rotated. Each pair may be not rotated at all, rotated clockwise by one position, rotated clockwise by two positions or rotated clockwise by three positions. As mentioned in the previous paragraph, any other move on a pair of dials has the same effect on the dials as one of these four moves. Therefore, there are a total of four possible moves that can be performed on each of the $k-1$ pairs of dials. This means any final configuration that can be achieved by a sequence of moves can be achieved by one of these 4^{k-1} sets of moves.

It is possible that two sets of moves lead to the same final configuration, so this merely establishes that there are at most 4^{k-1} attainable configurations.

Suppose $a_1, a_2, \ldots, a_{k-1}$ is a sequence with a_r taking one of the values $1, 2, 3$, or 4 for each $1 \leq r \leq k-1$. There are 4^{k-1} such sequences. By rotating the first two dials, we can make the number on top of the first dial equal to a_1. This will also rotate the second dial, but we can now rotate the second and third dials to get the second dial to have a_2 at the top. Furthermore, this will not change the number at the top of the first dial. Continuing in this way, we can rotate the third and fourth dials to get the third dial to have a_3 at the top without changing the first two dials. This can be continued to get a_r on the top of dial r for $1 \leq r \leq k-1$. Regardless of the position of the k^{th} dial, this gives a way to produce at least 4^{k-1} different configurations since any two of these configurations differ on at least one of the first $k-1$ dials.

Therefore, there are at least 4^{k-1} configurations. Since there are also at most 4^{k-1} configurations, this means there are exactly 4^{k-1} configurations.

(b) As in part (a), the order in which moves are applied does not matter. It only matters which moves are applied.

There are $k-1$ possible pairs of dials to be rotated and n possible positions by which to rotate each pair. As in part (a), the effect of any number of rotations of the same pair of dials can be achieved by a single clockwise rotation. Following a similar argument to the one in part (a) (with 4 replaced by n), this means there are no more than n^{k-1} possible configurations.

Also similar to part (a), the first $k-1$ dials can be configured in any of the n^{k-1} possible ways (though without control over the k^{th} dial). This shows that there are at least n^{k-1} configurations.

Since there are also at most n^{k-1} attainable configurations, there must be exactly n^{k-1} attainable configurations.

(c) We first suppose k is even. Consider the following sequence of moves, each by one position:

Dials	Direction
1 and 2	clockwise
2 and 3	counterclockwise
3 and 4	clockwise
4 and 5	counterclockwise
$\vdots$	$\vdots$
$k-2$ and $k-1$	counterclockwise
$k-1$ and k	clockwise

This alternating pattern ends up with the final move being clockwise because k is even. Each of dials 2 through $k-1$ has been rotated by one position clockwise and one position counterclockwise. Therefore, the net effect of this sequence of moves is to rotate dials 1 and k clockwise by one position and to leave all other dials in their original position. This sequence of moves could be repeated to attain any configuration achievable by the new type of move. Therefore, the new type of move can be mimicked by the original moves, so including it does not introduce any new attainable configurations.

Thus, when k is even, the number of configurations is n^{k-1}.

We now suppose k and n are odd and perform an "alternating" sequence of moves like in the previous case. Again, each move is by one position:

Dials	Direction
1 and 2	clockwise
2 and 3	counterclockwise
3 and 4	clockwise
4 and 5	counterclockwise
$\vdots$	$\vdots$
$k-2$ and $k-1$	clockwise
$k-1$ and k	counterclockwise
k and 1	clockwise

Since k is odd, this alternating pattern leads to dials $k-1$ and k being rotated counterclockwise, and finally dials k and 1 being rotated clockwise. The net effect of this sequence of moves is that dial 1 is rotated clockwise by two positions and all other dials are unchanged.

Since n is odd, $n+1$ is even, which means $\frac{n+1}{2}$ is an integer. If the sequence of moves above is repeated $\frac{n+1}{2}$ times, the net effect will be to rotate the first dial by $2\binom{n+1}{2} = n+1$ positions clockwise, which is the same as rotating by one position clockwise.

By repeating everything that has been done so far, we can independently move the first dial to any position we like without changing the position of any other dials.

Now imagine performing a new sequence of moves similar to those in the table above by increasing each integer in the above table by 1, with the exception of k which we change to 1. This modified sequence of moves will rotate dial 2 by two positions clockwise and have no overall effect on any other dial. Using the same reasoning as in the previous paragraph, this modified sequence can be repeated $\frac{n+1}{2}$ times to have the effect of rotating dial 2 clockwise by one position. Repeating all of this as many times as desired, we can achieve the effect of rotating dial 2 by any number of positions without rotating any other dials.

This can be repeated for any other dial. In other words, with the new type of move, each dial can be moved to any position without changing any of the others.

Therefore, when k and n are both odd, each of the n^k configurations of the dials is attainable.

Finally, we consider the case when k is odd and n is even. As with the case when k and n are both odd, it is still possible to rotate any individual dial by exactly two positions clockwise without changing the position of any other dial. If you read that part of the argument closely, you will notice that it only relied on the fact that k was odd and was independent of n.

Similar to the argument used for parts (a) and (b), by only using the original allowed moves, we can get the dials to a configuration where the first $k-1$ dials are in any positions we like. Furthermore, since we can rotate the k^{th} dial by multiples of two positions clockwise, there are $\frac{n}{2}$ positions of the k^{th} dial for each of the n^{k-1} configurations of the first $k-1$ dials. This means there are at least

$$\frac{n}{2}\left(n^{k-1}\right) = \frac{n^k}{2}$$

attainable configurations.

We will now argue that there are at most $\frac{n^k}{2}$ configurations.

Notice that in the initial configuration, the sum of the numbers showing at the top of each dial is $k \times 1 = k$ which we are assuming is odd. Each move changes two dials by the same amount, which means it adds or subtracts an even number from the total of the numbers on the top of the dials. Therefore, every attainable configuration must have the property that the total of the numbers showing on the top of the dials is odd.

Since n is even, exactly $\frac{n}{2}$ of the integers in the list $1, 2, 3, 4, \ldots, n$ are even, and exactly $\frac{n}{2}$ are odd. Suppose the dials are configured in some way and for each r with $1 \leq r \leq k-1$, the number showing at the top of the r^{th} dial is a_r. If $a_1 + a_2 + \cdots + a_{k-1}$ is even, then turning the k^{th} dial so that any of the $\frac{n}{2}$ numbers $1, 3, 5, \ldots, n-1$ is at the top will make the sum of the numbers of the top of all k dials odd. Turning the k^{th} dial to any of the other $\frac{n}{2}$ positions will make the total even. Similarly, if $a_1 + a_2 + \cdots + a_{k-1}$ is odd, then setting the k^{th} dial to have any of $2, 4, 6, \ldots, n$ at the top will make the total odd, and setting it to any of $1, 3, 5, 7, \ldots, n-1$ will make the total odd.

Either way, for any of the n^{k-1} configurations of the first $k-1$ dials, there are $\frac{n}{2}$ ways to arrange the k^{th} dial to make the sum of the numbers at the top of the dials odd, and $\frac{n}{2}$ ways to make the sum even. This means there are at least $\frac{n^k}{2}$ configurations with the sum of the top numbers being odd, and at least $\frac{n^k}{2}$ with the sum of the top numbers being even. Since $\frac{n^k}{2} + \frac{n^k}{2} = n^k$, this accounts for all configurations of the k dials. Therefore, there are exactly $\frac{n^k}{2}$ configurations of the dials for which the sum of the entries at the top of the dials is odd.

This implies there are at most $\frac{n^k}{2}$ attainable configurations, so there are exactly $\frac{n^k}{2}$ attainable configurations. The following table summarizes the results we have collected:

k	n	# of attainable configurations
even	even	n^{k-1}
even	odd	n^{k-1}
odd	even	$\dfrac{n^k}{2}$
odd	odd	n^k