Problem of the Month

Problem 3: December 2019

Problem

Let a, b, c, and d be rational numbers and $f(x) = ax^3 + bx^2 + cx + d$. Suppose $f(n)$ is an integer whenever n is an integer and that

$$\frac{1}{3}n^3 - n - \frac{2}{3} \leq f(n) \leq \frac{1}{3}n^3 + n^2 + 2n + \frac{4}{3}$$

for every integer n with the possible exception of $n = -2$.

(a) Show that $a = \frac{1}{3}$.

(b) Find $f(10^{2019}) - f(10^{2019} - 1)$.

Hint

The graphs of the polynomials $y = \frac{1}{3}x^3 - x - \frac{2}{3}$ and $y = \frac{1}{3}x^3 + x^2 + 2x + \frac{4}{3}$ look quite different near the origin. However, since they have the same leading term, they look nearly identical if you zoom out. Try graphing these two cubics on the same axes using graphing software. The function $f(x)$ lies between these two cubic functions (at least on integer inputs), so it should have the same overall "shape". This suggests that $a = \frac{1}{3}$. There are short arguments to justify this using limits, but there are also more elementary approaches. One thing you might try is to subtract $\frac{1}{3}n^3$ from each of the three expressions in the chain of inequalities. If $a \neq \frac{1}{3}$, you will have a cubic that is trapped between two quadratics, which should make you suspicious.

For (b), if you substitute $n = 0$ into the inequality, you should find that $-\frac{2}{3} \leq f(0) \leq \frac{4}{3}$. Since $f(0)$ is an integer, this means either $f(0) = 0$ or $f(0) = 1$. It is possible to find b, c, and d by gathering and using similar information.