Problem of the Month

Solution to Problem 4: January 2020

Solution

For this solution, it will be useful to have some notation involving remainders. Given an integer n and a positive integer d, the *remainder when n is divided by d* is defined to be the unique integer r in the list $0, 1, 2, \ldots, d-1$ with the property that $n - r$ is a multiple of d. For example, the remainder when 7 is divided by 3 is 1. This is because $7 - 1 = 6$ which is a multiple of 3, and 1 is the only number among $0, 1, 2$, that has this property. We will denote by $\text{rem}_d(n)$ the remainder when n is divided by d. Using this notation, the above example becomes, $\text{rem}_3(7) = 1$. Here is a useful fact that will be used several times:

$$\text{rem}_d(m+n) = \begin{cases} \text{rem}_d(m) + \text{rem}_d(n) & \text{if } \text{rem}_d(m) + \text{rem}_d(n) < d \\ \text{rem}_d(m) + \text{rem}_d(n) - d & \text{otherwise.} \end{cases}$$

Trying to convince yourself of this fact is a good way to gain some comfort with this notation.

(a) Suppose n is an integer that is not a multiple of 3. This means that either $\text{rem}_3(n) = 1$ or $\text{rem}_3(n) = 2$.

If $\text{rem}_3(n) = 1$, then $n = 3k + 1$ for some integer k. Then

$$n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1,$$

which means $\text{rem}_3(n^2) = 1$.

If $\text{rem}_3(n) = 2$, then there is some integer k satisfying $n = 3k + 2$. In this case,

$$n^2 = (3k+2)^2 = 3(3k^2 + 4k + 1) + 1,$$

which means $\text{rem}_3(n^2) = 1$.

If n is a multiple of 3, then n^2 is also a multiple of 3. In other words, if $\text{rem}_3(n) = 0$, then $\text{rem}_3(n^2) = 0$ as well. Since $\text{rem}_3(n)$ must be either $0, 1$, or 2, we have shown for any integer n that $\text{rem}_3(n^2) = 0$ when n is a multiple of 3, and $\text{rem}_3(n^2) = 1$ otherwise.

We will assume that a, b, and c are integers and are the edge lengths of a rectangular prism. We will also assume that at least two of a, b and c are not multiples of 3. We lose no generality by assuming that neither a nor b is a multiple of 3. From the previous paragraph, this means $\text{rem}_3(a^2) = 1$ and

$\text{rem}_3(b^2) = 1$. Using the fact before the beginning of the solution to part (a), we then have $\text{rem}_3(a^2 + b^2) = 2$. We showed in the beginning of this solution that the remainder when a perfect square is divided by 3 must be either 0 or 1. Since $\text{rem}_3(a^2 + b^2) = 2$, $a^2 + b^2$ cannot be a perfect square. By the Pythagorean Theorem, the length of the diagonal of the face with edge lengths a and b is $\sqrt{a^2 + b^2}$. Since $a^2 + b^2$ is not a perfect square, this diagonal length is not an integer and so the prism is not appealing.

Therefore, for a rectangular prism to be appealing, at least two of its edge lengths must be multiples of 3.

(b) We suppose a, b, and c are the edge lengths of an appealing prism and therefore the volume of the prism is abc. By part (a), we have that at least two of a, b, and c are multiples of 3, so the volume of the prism must be a multiple of 9. Note that $1584 = 9 \times 11 \times 16$ and since none of 9, 11, and 16 have any prime factors in common, it suffices to show that the volume must be a multiple of 11 and a multiple of 16. For the former, we will show that one of the edge lengths must be a multiple of 11. For the latter, we will show that at least two edge lengths must be multiples of 4.

Assume that a, b, and c are the edge lengths of an appealing prism and that none of a, b, and c is a multiple of 11. From this assumption we will deduce a contradiction. This will allow us to conclude that at least one of the edge lengths must be divisible by 11. Keep in mind that since the prism is appealing, we have that $a^2 + b^2$, $a^2 + c^2$, and $b^2 + c^2$ are all perfect squares by the Pythagorean Theorem.

By examining the eleven cases $\text{rem}_{11}(n) = 0$, $\text{rem}_{11}(n) = 1$, and so on up to $\text{rem}_{11}(n) = 10$, it can be shown that if n is an integer, then $\text{rem}_{11}(n^2)$ is in the list $0, 1, 3, 4, 5, 9$. Furthermore, the only way $\text{rem}_{11}(n^2) = 0$ is if $\text{rem}_{11}(n) = 0$ (n is a multiple of 11).

From this, we can examine the possible values of $\text{rem}_{11}(a^2 + b^2)$ for the various values of $\text{rem}_{11}(a^2)$ and $\text{rem}_{11}(b^2)$. Remember that we are assuming neither a nor b is a multiple of 11. The table below shows the value of $\text{rem}_{11}(a^2 + b^2)$ for every possible pair of nonzero values of $\text{rem}_{11}(a^2)$ and $\text{rem}_{11}(b^2)$. For example, the shaded cell indicates that $\text{rem}_{11}(a^2 + b^2) = 1$

when $\text{rem}_{11}(a^2) = 3$ and $\text{rem}_{11}(b^2) = 9$.

		$\text{rem}_{11}(a^2)$				
		1	3	4	5	9
$\text{rem}_{11}(b^2)$	1	2	4	5	6	10
	3	4	6	7	8	1
	4	5	7	8	9	2
	5	6	8	9	10	3
	9	10	1	2	3	7

Since $a^2 + b^2$ is a perfect square, we must have $\text{rem}_{11}(a^2 + b^2)$ equal to one of $0, 1, 3, 4, 5, 9$. Looking at the table, this means $\big(\text{rem}_{11}(a^2), \text{rem}_{11}(b^2)\big)$ must be one of:

$$(1,3), (3,1), (1,4), (4,1), (3,9), (9,3), (4,5), (5,4), (5,9), (9,5).$$

By similar reasoning, $\big(\text{rem}_{11}(a^2), \text{rem}_{11}(c^2)\big)$ and $\big(\text{rem}_{11}(b^2), \text{rem}_{11}(c^2)\big)$ must also be in the list above since $a^2 + c^2$ and $b^2 + c^2$ are perfect squares.

Next, we will rule out the possibility that any of $\text{rem}_{11}(a^2), \text{rem}_{11}(b^2)$, and $\text{rem}_{11}(c^2)$ is 1 or 3.

If $\text{rem}_{11}(a^2) = 1$, then upon examination of the list, it must be the case that $\text{rem}_{11}(b^2)$ and $\text{rem}_{11}(c^2)$ are both either 3 or 4. This would mean $\big(\text{rem}_{11}(b^2), \text{rem}_{11}(c^2)\big)$ is one of $(3,3), (3,4), (4,3)$, or $(4,4)$. None of these pairs are in the list, so we conclude that $\text{rem}_{11}(a^2) \neq 1$. By similar reasoning, we also have $\text{rem}_{11}(b^2) \neq 1$ and $\text{rem}_{11}(c^2) \neq 1$.

If $\text{rem}_{11}(a^2) = 3$, then $\text{rem}_{11}(b^2) = \text{rem}_{11}(c^2) = 9$ (remember, we have ruled out the possibility that either of these remainders is 1). This would mean $\big(\text{rem}_{11}(b^2), \text{rem}_{11}(c^2)\big) = (9,9)$, which is not in the list. Therefore, $\text{rem}_{11}(a^2) \neq 3$, and by symmetry, $\text{rem}_{11}(b^2) \neq 3$ and $\text{rem}_{11}(c^2) \neq 3$.

Therefore, none of $\text{rem}_{11}(a^2)$, $\text{rem}_{11}(b^2)$, or $\text{rem}_{11}(c^2)$ can be 1 or 3, so each of them must be in the list $4, 5, 9$.

None of $(4,4), (5,5)$ and $(9,9)$ are in the list above, so these remainders must all be different. Without loss of generality, we can assume that $\text{rem}_{11}(a^2) = 4, \text{rem}_{11}(b^2) = 5$, and $\text{rem}_{11}(c^2) = 9$. This means $\big(\text{rem}_{11}(a^2), \text{rem}_{11}(c^2)\big) = (4,9)$ which is not in the list. We have now run into a problem: if a, b, and c are the edge lengths of an appealing prism and none of a, b, and c is a multiple of 11, then there are no possible values of $\text{rem}_{11}(a^2), \text{rem}_{11}(b^2)$, and $\text{rem}_{11}(c^2)$.

Therefore, at least one of the edges lengths of an appealing prism is a multiple of 11.

We now prove that at least two edge lengths are multiples of 4. To do this, we first prove the following fact about Pythagorean Triples: If x, y, and z are integers with $x^2 + y^2 = z^2$, then at least one of x and y is even. It can be checked using similar analysis to part (a) that if n is an integer, then $\text{rem}_4(n^2) = 0$ if n is even and $\text{rem}_4(n^2) = 1$ if n is odd. Suppose x and y are odd integers. Then $\text{rem}_4(x^2 + y^2) = 1 + 1 = 2$, which is neither 0 nor 1, so $x^2 + y^2$ cannot be a perfect square. Therefore, if $x^2 + y^2$ is a perfect square, then at least one of x and y must be even.

Since the prism is appealing, we have that $a^2 + b^2$, $a^2 + c^2$, and $b^2 + c^2$ are perfect squares by the Pythagorean Theorem. The above fact then implies that at least one of a and b is even, at least one of a and c is even, and at least one of b and c is even. This will not happen if two of a, b, and c are odd, so we conclude that at least two of a, b, and c are even.

Suppose x and y are even integers with $\sqrt{x^2 + y^2}$ an integer. Then $\sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)}$ is also an integer. To see this, suppose z is a positive integer satisfying $x^2 + y^2 = z^2$. Since x and y are even, then so are x^2 and y^2, so $x^2 + y^2 = z^2$ must also be even. The only way for this to happen is for z to be even as well. Therefore,

$$\sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2} = \sqrt{\frac{z^2}{4}} = \frac{z}{2}$$

which is an integer. It follows that if all three of a, b, and c are even, then the prism with edge lengths $\frac{a}{2}, \frac{b}{2}$, and $\frac{c}{2}$ also has integer length diagonals, so it is also appealing. By what we have shown, this means at least two of $\frac{a}{2}, \frac{b}{2}$, and $\frac{c}{2}$ are even, so at least two of a, b, and c are multiples of 4.

We have now shown that if all three edge lengths are even, then at least two are multiples of 4.

We still know that at least two edge lengths are even, so we now assume exactly one of a, b, and c is odd. There is nothing special about how the edge lengths are labelled, so we suppose without loss of generality that a is odd and b and c are even. Suppose b is not a multiple of 4. This means $b = 2d$ for some odd integer d. Since a and d are both odd, there are

integers k and ℓ so that $a = 2k+1$ and $d = 2\ell+1$. Then

$$\begin{aligned} a^2+b^2 &= a^2+(2d)^2 \\ &= (2k+1)^2+(4\ell+2)^2 \\ &= 4k^2+4k+1+16\ell^2+16\ell+4 \\ &= 4[k(k+1)+4\ell^2+4\ell]+5. \end{aligned}$$

Notice that the quantity $k(k+1)$ is the product of consecutive integers, so it is even. Also, $4\ell^2+4\ell$ is even, which means the quantity $4[k(k+1)+4\ell^2+4\ell]$ must be a multiple of 8. Therefore, $\text{rem}_8(a^2+b^2) = 5$. It can be checked using similar reasoning to that in the earlier parts that $\text{rem}_8(n^2)$ is always either $0, 1$, or 4, which means a^2+b^2 cannot be a perfect square. This contradicts the fact that the prism is appealing, which means our additional assumption that b is not a multiple of 4 must have been false. In other words, b must be a multiple of 4. A similar argument shows that c is a multiple of 4. Therefore, if a is odd, then b and c are both multiples of 4.

(c) Observe that all three edge lengths of an appealing prism must be different. This is because if two edge lengths were both equal to some integer n, then the diagonal length would be $\sqrt{n^2+n^2} = \sqrt{2}n$ which is not an integer. Therefore, let us assume the edge lengths of an appealing prism are $44, b$, and c where $44 < b < c$. By the Pythagorean Theorem, there must be integers n and m with $44^2+b^2 = m^2$ and $44^2+c^2 = n^2$. Rearranging and factoring,

$$\begin{aligned} 44^2 &= m^2-b^2 &&= (m-b)(m+b) \\ 44^2 &= n^2-c^2 &&= (n-c)(n+c). \end{aligned}$$

The first equation implies $m-b$ and $m+b$ are a divisor pair for 44^2. There are eight ways to factor 44^2 into a product of two positive integers: 1×1936, 2×968, 4×484, 8×242, 11×176, 16×121, 22×88, and 44×44. Also, $0 < m-b < m+b$, which eliminates 44×44, so the pair $(m-b, m+b)$ must equal one of

$$(1,1936), (2,968), (4,484), (8,242), (11,176), (16,121), (22,88).$$

Notice also that $(m-b)+(m+b) = 2m$ which is even, so $m-b$ and $m+b$ are either both even or both odd. This reduces the list above to

$$(2,968), (4,484), (8,242), (22,88).$$

Since

$$m = \frac{(m-b)+(m+b)}{2} \quad \text{and} \quad b = \frac{(m+b)-(m-b)}{2},$$

if $(m-b, m+b) = (2, 968)$, we get that $m = 485$ and $b = 483$. The other three possibilities lead to $(b, m) = (240, 244)$, $(b, m) = (117, 125)$, and $(b, m) = (33, 55)$. Therefore, the possibilities for b are $33, 117, 240$, and 483. The same analysis applied to $44^2 = (n-c)(n+c)$ shows that c is also one of $33, 117, 240$, and 483. This gives 16 possibilities for the pair (b, c). However, we are assuming $b < c$ and it follows from part (c) that b and c cannot both be odd. This leads to just three possibilities for the pair (b, c), which are

$$(33, 240), (117, 240), (240, 483).$$

One can now check by hand or by calculator that $33^2 + 240^2 = 58\,689$ is not a perfect square and that $240^2 + 483^2 = 290\,889$ is not a perfect square, but $117^2 + 240^2 = 267^2$. Therefore, there is an appealing prism with edge lengths $44, 117$ and 240. This was discovered by Paul Halcke in 1719.

<u>Further Discussion</u>: It can actually be verified with some effort that 44 is the smallest edge length that can occur as the edge length of an appealing prism. For example, it isn't too much work to show that the shortest edge length of an appealing prism cannot be prime. Slightly more work shows that the shortest edge length cannot be twice a prime. This rules out many of the numbers from 1 through 43 as the shortest edge length of an appealing prism, though there are still many cases to check in a way similar to the solution above.
Our name *appealing prism* was actually a disguise for the name *Euler Brick*. You might want to search this on the internet. It is also a nice exercise to program a computer to generate more Euler Bricks and help you notice other patterns about the prime factors of their edge lengths. An Euler Brick with an integer-length space diagonal is called a *Perfect Cuboid*. In fact, determining whether or not a Perfect Cuboid exists is an open problem! That is, nobody has ever produced one, and nobody has ever managed to prove that none exist! This is an example of an intriguing problem in *number theory* that is quite easy to state, but apparently very difficult to solve. According to Wikipedia, it has been verified that there are no perfect cuboids having a edge length less than 5×10^{11}.