

Problem of the Month

Problem 7: April 2020

Define a function f whose input and output are both lists of n nonnegative integers by

$$f(a_1, a_2, \dots, a_n) = (|a_1 - a_2|, |a_2 - a_3|, \dots, |a_{n-1} - a_n|, |a_n - a_1|)$$

where, as usual, $|x|$ represents the absolute value of x .

For example, $f(1, 2, 3, 4) = (|1 - 2|, |2 - 3|, |3 - 4|, |4 - 1|) = (1, 1, 1, 3)$ and $f(2, 3, 5) = (|2 - 3|, |3 - 5|, |5 - 2|) = (1, 2, 3)$.

We will denote by f^k the function that *iterates* the application of f a total of k times. For example,

$$f^4(1, 1, 1, 3) = f^3(0, 0, 2, 2) = f^2(0, 2, 0, 2) = f(2, 2, 2, 2) = (0, 0, 0, 0).$$

We will call the list (a_1, a_2, \dots, a_n) *smooth* if there is some m for which $f^m(a_1, \dots, a_n)$ is the list of n zeros. That is, a list is smooth if some number of applications of f will result in the list of all zeros. For example, $(1, 1, 1, 3)$ is smooth since $f^4(1, 1, 1, 3) = (0, 0, 0, 0)$, as demonstrated above.

- (a) Find lists of length 5 and 7 that are not smooth.
- (b) Show that for all odd integers $n \geq 1$ there exists a list L of length n that is not smooth.
- (c) How many smooth lists (a, b, c) are there with a , b , and c each no larger than 100?
- (d) Suppose L is a list of length 4 consisting of only zeros and ones. Show that L is smooth.
- (e) Show that all lists of length 4 are smooth.