



Problem of the Month

Solution to Problem 8: May 2020

Solution

Throughout this solution, we will refer to the lattice points closest to the origin.

By the Pythagorean theorem, the distance from the point (a, b) to the origin is $\sqrt{a^2 + b^2}$. If a and b are both integers, the smallest $\sqrt{a^2 + b^2}$ can be is 0, occurring exactly when $a = b = 0$. The next smallest that $\sqrt{a^2 + b^2}$ can be is 1, and this occurs when one of a and b is ± 1 and the other is 0. Thus, there are four lattice points at a distance of 1 from the origin, and they are $(-1, 0)$, $(0, -1)$, $(0, 1)$, and $(1, 0)$. We will refer to the set of these four points as S . The next smallest that $\sqrt{a^2 + b^2}$ can be when a and b are integers is $\sqrt{2}$, and this occurs at the four points $(-1, -1)$, $(-1, 1)$, $(1, -1)$, and $(1, 1)$. We will call this set of points T . Continuing in this way, the table below catalogues the lattice points no more than $\sqrt{8}$ units from the origin and names certain sets of four of these points.

$d = \sqrt{a^2 + b^2}$	Approximate value of d	Possible pairs (a, b)	
0	0	$(0, 0)$	
1	1	$(-1, 0)$, $(0, -1)$, $(0, 1)$, $(1, 0)$	S
$\sqrt{2}$	1.4142	$(-1, -1)$, $(-1, 1)$, $(1, -1)$, $(1, 1)$	T
2	2	$(-2, 0)$, $(0, -2)$, $(0, 2)$, $(2, 0)$	U
$\sqrt{5}$	2.2361	$(-2, 1)$, $(-1, -2)$, $(1, 2)$, $(2, -1)$	V
		$(-2, -1)$, $(-1, 2)$, $(1, -2)$, $(2, 1)$	W
$\sqrt{8}$	2.8284	$(-2, -2)$, $(-2, 2)$, $(2, -2)$, $(2, 2)$	X

Table 1

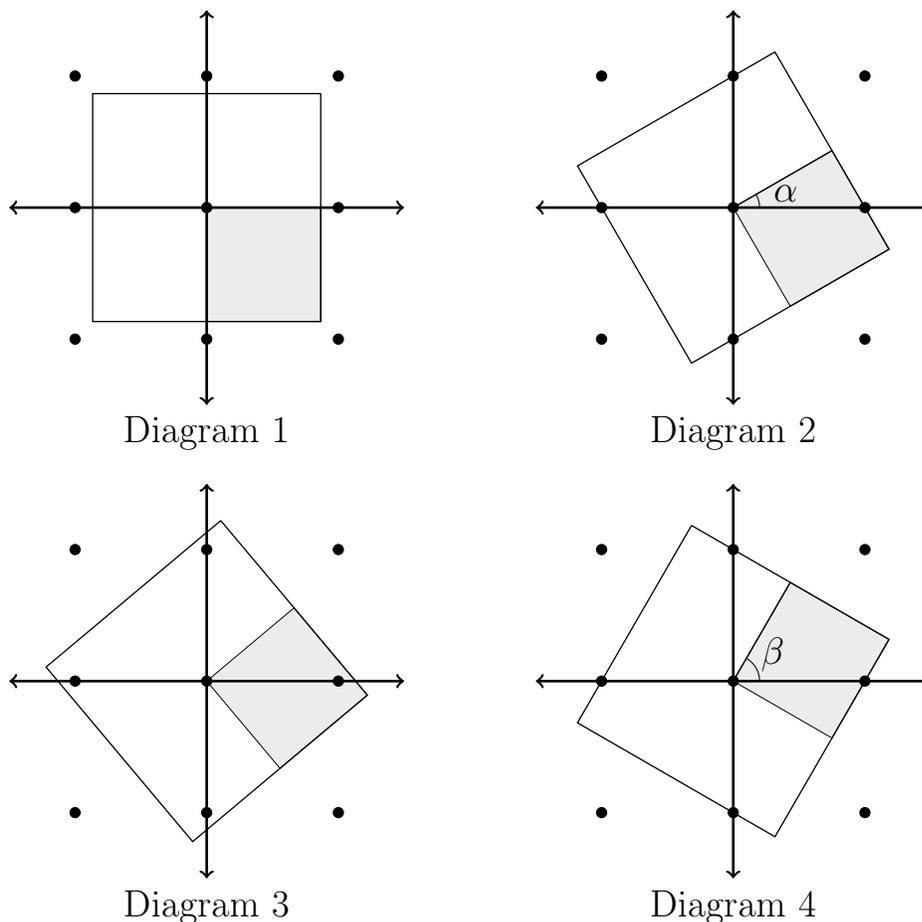
In part (d), we will use a geometric notion called *convexity*. A polygon in the plane is called *convex* if for every pair of points inside the polygon, the line segment connecting these two points is entirely inside the polygon. Another way to think of this is that every point inside a convex polygon can “see” every other point.

For example, a square is convex because if any two points inside a square are joined by a line segment, the entire line segment will be inside the square. In fact, all regular polygons are convex. Here is a fact we will need later: Suppose P and Q are convex polygons and that the perimeter of P is entirely inside Q . Then all of P lies inside Q .

- (a) In general, a square of radius r centred at the origin is completely inside the circle of radius r centred at the origin. Thus, a point that is inside a square of radius r centred at the origin must also be inside the circle of radius r centred at the origin, and hence, cannot be more than r units from the origin. Using Table 1 and observing that $\frac{\sqrt{6}}{2} \approx 1.225$, this means the only lattice points that are close enough to the origin to possibly be inside a square of radius $\frac{\sqrt{6}}{2}$ centred at the origin are the five points $(0,0)$, $(-1,0)$, $(0,-1)$, $(0,1)$, and $(1,0)$. Thus, if there are five points inside a square of radius $\frac{\sqrt{6}}{2}$ centred at the origin, the points must be the origin and the points in S (Table 1).

We will first discuss how to find the probability that $(1,0)$ is inside the square. Next, we will discuss how this probability is equal to the probability that there are five points inside the square, and finally, we will compute the probability.

To determine the probability that $(1,0)$ is inside the square, we consider the following four diagrams showing the square in various positions. In all four diagrams, the shaded quadrant represents the same quadrant of the square. The angles α and β are in degrees.



In Diagram 1, the sides of the square are parallel/perpendicular to the axes so that the midpoints of the four sides lie on the axes. In Diagram 2, the

square has been rotated an angle of α counterclockwise so that $(1, 0)$ lies on the perimeter of the square. In Diagram 3, the square has been rotated a bit more in the counterclockwise direction so that $(1, 0)$ is in the interior of the square. In Diagram 4, the square has been rotated a total angle of $\beta > \alpha$ counterclockwise so that $(1, 0)$ lies on the perimeter again. From the position where the square is rotated α to the position where it is rotated β , the point $(1, 0)$ is inside the square, and in fact, is inside the shaded region. This means the angle through which we can rotate the square so that $(1, 0)$ is inside the shaded region has a measure of $\beta - \alpha$.

Therefore, the probability that $(1, 0)$ is inside the shaded region is $\frac{\beta - \alpha}{360^\circ}$ since 360° represents one complete rotation of the square around the origin, during which the square will be in every possible position.

By symmetry, the probability that $(1, 0)$ is in each of the other three quadrants of the square is the same, so the probability that $(1, 0)$ is inside the square is

$$\frac{4(\beta - \alpha)}{360^\circ} = \frac{\beta - \alpha}{90^\circ}.$$

Notice that in Diagrams 2 and 4 above, not only is $(1, 0)$ on the perimeter of the square, all four points in S appear to be on the perimeter. As well, in Diagram 3, all four points in S are inside the square. This is due to the 90° -rotational symmetry of both the square and the points in S . In other words, it is impossible to have $(1, 0)$ inside the square without having all four points in S inside the square. You may want to spend some time thinking about this idea. The origin is always inside the square, so the probability that there are five lattice points inside the square is equal to the probability that $(1, 0)$ is inside the square.

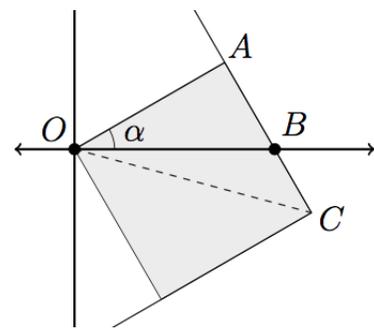
Thus, to compute the probability that there are five lattice points inside the square, we will compute the difference $\beta - \alpha$ and divide by 90° .

The picture to the right is the right side of Diagram 2 with several points labelled.

Since O is at the centre of the square and A is the midpoint of a side of the square, the shaded region is a square. This means $AO = AC$ and $\angle CAO = 90^\circ$. By the Pythagorean theorem, $AO^2 + AC^2 = OC^2$ so $2AO^2 = \left(\frac{\sqrt{6}}{2}\right)^2 = \frac{6}{4}$ which simplifies to $AO^2 = \frac{3}{4}$.

Since $AO > 0$, this means $AO = \frac{\sqrt{3}}{2}$.

We have that $OB = 1$ since B has coordinates $(1, 0)$. Since $\angle BAO = 90^\circ$, $\cos \angle AOB = \frac{AO}{BO} = \frac{\sqrt{3}}{2}$. Since $\alpha < 90^\circ$, this implies $\alpha = 30^\circ$. A similar



calculation in Diagram 4 shows that the angle $90^\circ - \beta = 30^\circ$, so $\beta = 60^\circ$.

Thus, $\beta - \alpha = 60^\circ - 30^\circ = 30^\circ$. From the calculation above, this means the probability that there are five points inside the square is $\frac{30^\circ}{90^\circ} = \frac{1}{3}$.

- (b) From the beginning of the solution to part (a), there are only five lattice points within $\frac{\sqrt{6}}{2}$ units of the origin, so $f\left(\frac{\sqrt{6}}{2}\right) \leq 5$. Moreover, it is possible for a square of radius $\frac{\sqrt{6}}{2}$ centred at the origin to have five lattice points inside it, so $f\left(\frac{\sqrt{6}}{2}\right) \geq 5$. Together with $f\left(\frac{\sqrt{6}}{2}\right) \leq 5$, we have $f\left(\frac{\sqrt{6}}{2}\right) = 5$.
- (c) Consider the rotational symmetry mentioned in the solution to part (a). Regardless of the position of the square, if it is rotated 90° about the origin, it will land perfectly on top of itself. This means if a point (a, b) is inside the square, then the point $(-b, a)$ obtained by rotating (a, b) by 90° counter-clockwise will also be inside the square. Similarly, $(-a, -b)$ and $(b, -a)$ will be inside the square.

We consider the lattice points other than the origin as being broken into sets of four points. Each set of four points consists of a lattice point together with the three other lattice points that can be obtained by rotating it about the origin by a multiple of 90° . The sets S, T, U, V, W , and X labelled in Table 1 are examples of such sets of four points.

As discussed above, if we consider such a set of four points and *any* square centred at the origin, either all four points are inside the square or none of the four points are inside the square. Thus, the lattice points inside any square centred at the origin are the origin and some number (possibly zero) of complete sets of four points.

This means the number of lattice points inside *any* square centred at the origin is 1 more than a multiple of 4. It follows that $f(r)$ must be 1 more than a multiple of 4.

- (d) The answer is $r = \frac{4\sqrt{10}}{5}$. To show this, we need to verify that $f(r) = 17$ and that if $c < r$, then $f(c) \neq 17$. The latter will be done by showing that if $f(c) = 17$, then $c \geq r$.

First, we will show that $f(r) = 17$. Consider the quadrilateral in Diagram 5 obtained by intersecting the lines through the pairs of points $(-2, 1)$ and $(0, 2)$, $(1, 2)$ and $(2, 0)$, $(2, -1)$ and $(0, -2)$, and $(-1, -2)$ and $(-2, 0)$.

The sides of the quadrilateral have slopes $\frac{1}{2}$ and -2 which means opposite sides are parallel and adjacent sides are perpendicular, so the quadrilateral is a rectangle. The rectangle has 90° rotational symmetry since the points

defining its sides do. Thus, the rectangle is a square centred at the origin. To find its radius, we first find the coordinates of one of its vertices. The vertex in the first quadrant is the intersection of the line through $(-2, 1)$ and $(0, 2)$ and the line through $(1, 2)$ and $(2, 0)$. These lines have equations $y = \frac{1}{2}x + 2$ and $y = -2x + 4$, respectively, and intersect at $(\frac{4}{5}, \frac{12}{5})$. The radius of the square is the distance from this point to the origin, which is

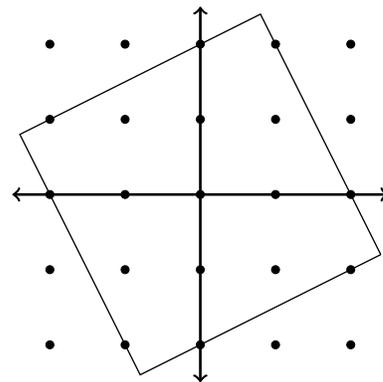


Diagram 5

$$\sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{12}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{144}{25}} = \frac{\sqrt{160}}{\sqrt{25}} = \frac{4\sqrt{10}}{5} = r.$$

Using the eight points defining the perimeter of the square and the *convexity* property from the first page of the solution, it isn't hard to verify that the 17 lattice points appearing to be inside the square (see Diagram 5) are indeed inside the square. This means $f(r) \geq 17$.

Since $r \approx 2.53$ which is less than $\sqrt{8} \approx 2.828$, the data in Table 1 implies $f(r) \leq 21$ since there are only 21 lattice points within r units of the origin.

From part (c), $f(r)$ must be one more than a multiple of 4, which implies $f(r) = 17$ or $f(r) = 21$. We will now rule out the possibility that $f(r) = 21$. As noted above, there are only 21 lattice points within r units of the origin, which means that if $f(r) = 21$, then there is some square of radius r centred at the origin that contains all 21 of these lattice points. Consider the irregular octagon obtained by connecting the lattice points at a distance of $\sqrt{5}$ or 2 from the origin, as shown in Diagram 6.

Suppose $f(r) = 21$. Then there must be a square of radius r containing the points defining the perimeter of this octagon. Since squares are convex, the entire perimeter of the octagon must be contained in the square. Since the octagon in Diagram 6 is convex (think about this), the fact before the solution to part (a) implies the entire octagon is inside the square.

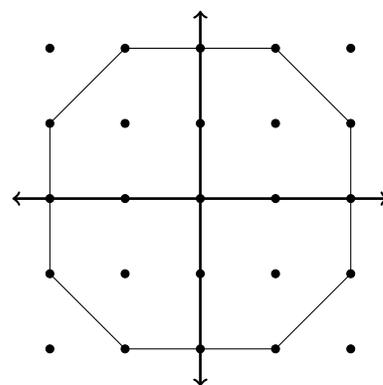


Diagram 6

However, it is easily checked that the area of the octagon is 14 units squared and the area of a

square of radius r is $2r^2 = \frac{2(16)(10)}{25} = \frac{64}{5} = 12.8$ units squared. Thus, the area of the octagon is larger than that of a square of radius r , so the octagon cannot be completely inside the square. Therefore, $f(r) \neq 21$, so $f(r) = 17$.

We will now show that if $c < r$, then $f(c) \neq 17$. This is equivalent to showing that if $f(c) = 17$, then $c \geq r$. Together with $f(r) = 17$, this means r is the smallest real number satisfying $f(r) = 17$.

We now assume that c is a real number with $f(c) = 17$. This means there is a square of radius c centred at the origin that has exactly 17 lattice points inside it. From this, we will deduce that $c \geq r$.

Showing that $c \geq r$ is equivalent to showing that c is not less than r . To do this, we assume that $c \leq r$ and deduce that $c = r$. You may want to consider the logic of this approach for a moment before reading on. Thus, we are now assuming that $c \leq r$ and that there is a square of radius c centred at the origin with 17 lattice points inside it. From this, we will deduce that $c = r$.

Since $c \leq r < \sqrt{8}$, the 17 lattice points inside the square of radius c must be among the 21 lattice points closer than $\sqrt{8}$ units to the origin. This means the 17 points inside the square must be comprised of the origin and some of the complete sets S, T, U, V , and W . If all points in V and W are inside the square, then the octagon from Diagram 6 is inside the square, which would mean $c > r$ by the previous argument. This means we cannot include both sets V and W . Since there are 17 lattice points inside the square, all points from S, T , and U are inside the square, as well as the points from exactly one of V and W . We assume that the 17 lattice points inside the square are the origin and those in the sets S, T, U , and V . If we replace V by W , the argument is similar.

In particular, the eight points defining the perimeter of the irregular octagon in Diagram 7 must all be inside the square. This octagon is convex, which means it is completely inside the square. Given the shape and position of this octagon, it seems reasonable that the smallest square containing it is the one in Diagram 5. To confirm this, it will be useful to find a point on the perimeter of this octagon at a minimum distance from the origin. By rotational symmetry, one such point will be either on the line segment L connecting $(-1, -2)$ to $(-2, 0)$ or M connecting $(-2, 0)$ to $(-2, 1)$. The segment L lies on the line with equation $y = -2x - 4$, which means a point on this segment takes the form $(x, -2x - 4)$. The distance from such a point to the origin is

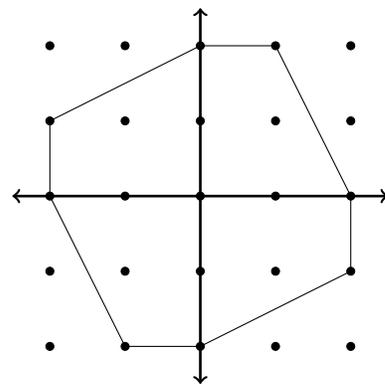


Diagram 7

$$\sqrt{x^2 + (-2x - 4)^2} = \sqrt{5x^2 + 16x + 16} = \sqrt{5\left(x + \frac{8}{5}\right)^2 + \frac{16}{5}}.$$

This quantity is minimized when the positive quantity inside the radical is

minimized, and this happens when $5\left(x + \frac{8}{5}\right)^2 = 0$ or $x = -\frac{8}{5}$. The distance is $\sqrt{0 + \frac{16}{5}} = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$.

On the other hand, a point on the vertical segment M has the form $(-2, y)$ for some y and has distance from the origin equal to

$$\sqrt{(-2)^2 + y^2} = \sqrt{4 + y^2}$$

which is minimized when $y = 0$. The distance when $y = 0$ is $\sqrt{4} = 2$ which is larger than $\frac{4\sqrt{5}}{5} \approx 1.7889$. Therefore, among points on the perimeter of the octagon in Diagram 7, the closest to the origin is at a distance of $\frac{4\sqrt{5}}{5}$.

We now consider a line segment connecting the origin to the midpoint of a side of the square. An application of the Pythagorean theorem shows that the length of this segment is $\frac{c}{\sqrt{2}}$. Since the octagon is completely inside the square, this line segment must intersect the perimeter of the octagon at some point. Therefore, if we let d be the distance from this point to the origin, we have $d \leq \frac{c}{\sqrt{2}}$. As shown above, the closest point *anywhere* on the perimeter of the octagon to the origin is at a distance of $\frac{4\sqrt{5}}{5}$, which means

$$\frac{4\sqrt{5}}{5} \leq d \leq \frac{c}{\sqrt{2}}$$

from which it follows that

$$r = \frac{4\sqrt{10}}{5} = \frac{4\sqrt{5}\sqrt{2}}{5} \leq c$$

where the inequality comes from rearranging $\frac{4\sqrt{10}}{5} \leq \frac{c}{\sqrt{2}}$. Together with our assumption that $c \leq r$, we have that $c = r$, which is what we set out to show.

Therefore, any square containing 17 lattice points has a radius greater than or equal to r , which means r is the smallest real number with $f(r) = 17$.