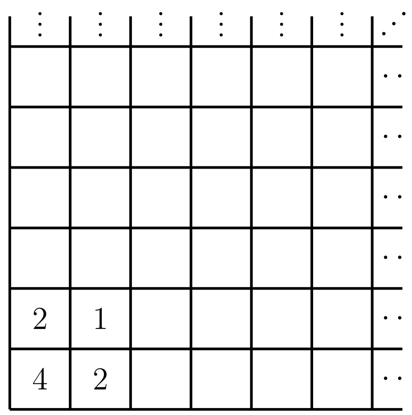
Problem of the Month
Problem 0: September 2020

Problem

A rectangular array extends up and to the right with infinitely many rows and infinitely many columns. Integers are placed in the four "bottom-left" cells as shown with 4 in the bottom-left corner, 2 in each of the cells sharing a side with the cell containing 4, and 1 in the cell immediately to the right and above the 4.



When referring to rows and columns, we start the enumeration from the bottom and the left. For example, the "third row" refers to the third row from the bottom.

Integers are placed in the remaining cells *recursively* as follows:

- In the first and second rows, each remaining cell contains the sum of the integer in the cell immediately to its left and twice the integer two cells to its left. For example, the third cell in the first row contains the integer $2 + 2(4) = 10$.
- Cells in or above the third row contain the sum of the integer in the cell immediately below and twice the integer in the cell two below. For example, the second cell in the third row contains the integer $1 + 2(2) = 5$.

We will denote by $f(m, n)$ the integer in the m^{th} row and the n^{th} column.

(a) Show that every cell other than those in the first two columns contains the sum of the integer in the cell immediately to its left and twice the integer in the cell two to its left. That is, show that $f(m, n) = f(m, n-1) + 2f(m, n-2)$ for all integers $m \geq 1$ and $n \geq 3$.

(b) Prove that $f(m, m)$ is a perfect square for every integer $m \geq 1$. In other words, prove that all of the cells on the diagonal contain perfect squares.

(c) Determine the value of $f(456, 789)$.

Hint

Before doing any of the parts of this problem, it is a good idea to fill in some of the table in order to gain some intuition. However, working out a few cases and convincing yourself something is true is not the same as writing a formal proof.

(a) By carefully using the rules defining the numbers in the cells, an informal argument can be given using a bit of algebra. However, to give a formal argument, it is highly recommended to use *mathematical induction*. You may want to read about this before trying to write a proof.

(b) Compute a few of the diagonal entries. Do you notice anything about what numbers were squared to get these diagonal entries? Once again, mathematical induction will be useful in formalizing any observations you make.

(c) Can you find a formula for the entries in the second row and second column?