Problem of the Month

Solution to Problem 0: September 2020

We will use the notation introduced in the problem statement and denote by $f(m, n)$ the integer in the cell in the m^{th} row and the n^{th} column. Using this notation, the example given in the first bullet point in the problem statement translates to $f(1, 3) = f(1, 2) + 2f(1, 1) = 2 + 2(4) = 10$.

Before attempting any of these problems, it is a good idea to fill in a bit more of the array in order to gain some intuition:

⋮	⋮	⋮	⋮	⋮	⋮	⋰
						⋯
34	17	85	119	289		⋯
14	7	35	49	119		⋯
10	5	25	35	85		⋯
2	1	5	7	17		⋯
4	2	10	14	34		⋯

There are several observations you might make at this point. For example, 4, 1, 25, 49, and 289 are all perfect squares, so the claim in part (b) might seem plausible. You might also notice that the array appears to be symmetric in the diagonal. That is, it seems $f(m, n) = f(n, m)$ for all positive integers m and n.

(a) Using our notation, the task is to show that

$$f(m, n) = f(m, n - 1) + 2f(m, n - 2) \tag{1}$$

for every pair of positive integers (m, n) with $n \geq 3$. For cells in the bottom two rows ($m = 1$ or $m = 2$), the integers are defined in this way. That is, Equation (1) holds when $m = 1$ and $m = 2$ by definition. Looking back at the partially-filled array above, you might want to check that this identity holds for a few other cells. For example, $f(4, 5) = 119$, $f(4, 4) = 49$, and $f(4, 3) = 35$ and indeed $119 = 49 + 2(35)$.

Now we note that we can use the fact that Equation (1) holds for cells in the first and second row (for $m = 1$ and $m = 2$) to show that Equation (1) also holds for cells in the third row (for $m = 3$). Consider the pair $(3, n)$ for some $n \geq 3$. We know that $f(3, n) = f(2, n) + 2f(1, n)$ by the rule in the second bullet point in the problem statement. Since Equation (1) holds for $m = 1$ and $m = 2$ we have $f(2, n) = f(2, n-1) + 2f(2, n-2)$ and $f(1, n) = f(1, n - 1) + 2f(1, n - 2)$. Putting these together we get

$$\begin{aligned} f(3, n) &= f(2, n) + 2f(1, n) \\ &= [f(2, n - 1) + 2f(2, n - 2)] + 2[f(1, n - 1) + 2f(1, n - 2)] \\ &= [f(2, n - 1) + 2f(1, n - 1)] + 2[f(2, n - 2) + 2f(1, n - 2)] \end{aligned}$$

Notice that $f(2, n-1)+2f(1, n-1) = f(3, n-1)$ and $f(2, n-2)+2f(1, n-2) = f(3, n-2)$, again by how the array is defined. This gives us that $f(3,n) = f(3,n-1) + 2f(3,n-2)$ which is Equation (1) for $m = 3$, and so the equation holds for the third row.

We could now proceed to argue that since Equation (1) holds for $m = 1$, $m = 2$ and $m = 3$ it must be the case that Equation (1) also holds for $m = 4$. You might find it useful to try to write down this argument yourself. (The argument will be very similar to the one presented above, and you should find that you only need to explicitly use the fact that Equation (1) holds for $m = 2$ and $m = 3$ in your argument.) Instead we proceed more generally to argue that if Equation (1) holds for $m = 1$, $m = 2$, and so on up to $m = r$ then Equation (1) must also hold for $m = r + 1$. The rough idea is to show that you can always get "the next row".

Suppose that $r \geq 2$ is an integer and that Equation (1) holds for the cells in the first r rows. That is, we assume Equation (1) holds for all $n \geq 3$ when $m = 1$, and when $m = 2$, and so on up to when $m = r$. This means we have the following:

Definition 1: Given a positive integer t, we have $f(m,t) = f(m-1,t) + 2f(m-2,t)$ for all $m \geq 3$. (This is by the rule in the second bullet point in the problem statement.)

Assumption 1: Given a positive integer $t \leq r$, we have $f(t,n) = f(t,n-1) + 2f(t,n-2)$ for all $n \geq 3$. (This is our assumption from above.) In particular, we are assuming that $f(r,n) = f(r,n-1) + 2f(r,n-2)$ and $f(r-1,n) = f(r-1,n-1) + 2f(r-1,n-2)$.

We will show that the above statements imply that the identity also holds for the cells in the $(r+1)^{\text{st}}$ row. Consider a cell in the $(r+1)^{\text{st}}$ row that is not in the first two columns. That is, consider the pair $(r+1,n)$ for some $n \geq 3$. We wish to show that Equation (1) holds for this pair, that is, $f(r+1,n) = f(r+1,n-1) + 2f(r+1,n-2)$. Here is the calculation, using the assumptions above:

$$\begin{aligned}
f(r+1,n) &= f(r,n) + 2f(r-1,n) && \text{(by Definition 1)} \\
&= \big[f(r,n-1) + 2f(r,n-2)\big] + 2\big[f(r-1,n-1) + 2f(r-1,n-2)\big] \\
& && \text{(by Assumption 1)} \\
&= f(r,n-1) + 2f(r-1,n-1) + 2f(r,n-2) + 4f(r-1,n-2) \\
&= \big[f(r,n-1) + 2f(r-1,n-1)\big] + 2\big[f(r,n-2) + 2f(r-1,n-2)\big] \\
&= f(r+1,n-1) + 2f(r+1,n-2) && \text{(by Definition 1)}
\end{aligned}$$

We have shown that if the identity holds in the first r rows, then it holds in the first $r+1$ rows. Since it holds in the first two rows, it holds in the first three rows. Since it holds in first three rows, it holds in the first four rows. This continues indefinitely to imply that the identity holds in every row. We have just used what is known as *strong induction.*

(b) You may have noticed that the entries in the second row are identical to those in the second column. This is true because the second row and the second column start with the same two integers (2 followed by 1), and all subsequent integers in each are determined in the same way (where each integer depends on the two integers before it). With our notation, this means that for every positive integer n we have $f(n,2) = f(2,n)$.

Something more subtle that you may have noticed is that for all positive integers m and n, we have $f(m,n) = f(m,2)f(2,n)$. You may want to go back to the array and check this for a few pairs (m,n). Using this, and the fact above that $f(n,2) = f(2,n)$ for every $n \geq 1$, we get that

$$f(n,n) = f(n,2)f(2,n) = f(2,n)^2$$

which establishes that $f(n,n)$ is a perfect square for every $n \geq 1$. In fact, it establishes that the integers on the diagonal are the squares of the integers in the second row (or second column).

We will finish the solution to part (b) with a somewhat informal explanation of why $f(m,n) = f(m,2)f(2,n)$ for all positive integers m and n. This fact will also be used in the solution to part (c). A formal proof of this fact can be found after the solution to part (c).

The main observation is that each row is a "scalar multiple" of the second row. To see this, we argue as follows: Suppose two sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ satisfy $a_n = a_{n-1} + 2a_{n-2}$ and $b_n = b_{n-1} + 2b_{n-2}$ for $n \geq 3$. This means each sequence is entirely determined by its first two terms. Also, both sequences satisfy the same recursive rule.

Now suppose there is some constant c so that $b_1 = ca_1$ and $b_2 = ca_2$, that is, a_1 and a_2 can be scaled by the same factor to get b_1 and b_2, respectively. Then

$$b_3 = b_2 + 2b_1 = ca_2 + 2ca_1 = c(a_2 + 2a_1) = ca_3,$$

which says that b_3 can be obtained by scaling a_3 by the same factor. Continuing, we also have

$$b_4 = b_3 + 2b_2 = ca_3 + 2ca_2 = c(a_3 + 2a_2) = ca_4.$$

This reasoning can be continued to show that $b_n = ca_n$ for every positive integer n. The reason this happens is because the way in which subsequent terms rely on previous terms is *linear* (a slightly different use of the word than you may be used to). Roughly speaking, linear in this context means the terms in the sequence are obtained by scaling some of the previous terms and adding the results together.

Now note that the first two columns in the array contain sequences of integers with the above properties where b_n corresponds to the first column, a_n corresponds to the second column, and the constant is $c = 2$. Using the reasoning above, we can deduce that the integers in the first column are each exactly twice the integer to their right (in the second column). Now consider the second row and the m^{th} row. We know that the first two terms in the second row are 2 and 1, in that order, and the first two terms in the m^{th} row must be $2f(m,2)$ and $f(m,2)$, in that order. In part (a), we showed that the sequence in the m^{th} row satisfies the same recursive definition as the one in the second row. This means that the m^{th} row and second row contain sequences of integers with the above properties with $c = f(m,2)$. It follows that the integers in the m^{th} row must all be c times their corresponding integer in the second row. In other words, we have $f(m,n) = cf(2,n) = f(m,2)f(2,n)$.

(c) Using the identity $f(m,n) = f(m,2)f(2,n) = f(2,m)f(2,n)$, we get

$$f(456, 789) = f(456, 2)f(2, 789) = f(2, 456)f(2, 789)$$

so we can find $f(456, 789)$ by finding $f(2, 456)$ and $f(2, 789)$ and taking their product.

Since the rest of the solution will focus on the entries in the second row, we will simplify notation and define $g(n)$ to be $f(2,n)$ for each integer $n \geq 1$. From the table at the beginning of the solution, we get $g(1) = 2$, $g(2) = 1$, $g(3) = 5$, $g(4) = 7$, $g(5) = 17$. Continuing to compute terms, it can be checked that $g(6) = 31$, $g(7) = 65$, $g(8) = 127$, $g(9) = 257$, and $g(10) = 511$.

While the pattern may have been difficult to detect before, it may be easier to guess from the sequence

$$2, 1, 5, 7, 17, 31, 65, 127, 257, 511$$

as these numbers are all 1 away from a power of 2. In fact, 2 is 1 more than 2^0, 1 is 1 less than 2^1, 5 is 1 more than 2^2, 7 is 1 less than 2^3, and so on. Following this reasoning, we can see that when $1 \leq n \leq 10$, we have that

$$g(n) = 2^{n-1} + (-1)^{n-1}.$$

Proceeding once more by strong induction, we can prove that this identity holds for all $n \geq 1$. Assume for some $r \geq 2$ that $g(k) = 2^{k-1} + (-1)^{k-1}$ for every k from 1 to r inclusive. In particular, this implies $g(r) = 2^{r-1} + (-1)^{r-1}$ and $g(r-1) = 2^{r-2} + (-1)^{r-2}$. We will call these equations **Equation (2)** and **Equation (3)** respectively. By the definition of the integers in the second row, we have $g(r+1) = g(r) + 2g(r-1)$ which we will refer to as **Equation (4)**. Then we have

$$\begin{aligned}
g(r+1) &= g(r) + 2g(r-1) && \text{(Equation (4))} \\
&= \left(2^{r-1} + (-1)^{r-1}\right) + 2\left(2^{r-2} + (-1)^{r-2}\right) && \text{(Equations (2) and (3))} \\
&= 2^{r-1} + (-1)^{r-1} + 2^{r-1} + 2(-1)^{r-2} \\
&= 2^{r-1} + 2^{r-1} + (-1)^{r-1} + 2(-1)^{r-2} \\
&= 2\left(2^{r-1}\right) + (-1)^{r-2}(-1+2) \\
&= 2^r + (-1)^{r-2}(1) \\
&= 2^r + (-1)^{r-2}(-1)^2 \\
&= 2^r + (-1)^r
\end{aligned}$$

where the calculation after the second two equations is just arithmetic using exponent laws. This means $g(r+1) = 2^{(r+1)-1} + (-1)^{(r+1)-1}$, so by strong induction, we have that $g(n) = f(2, n) = 2^{n-1} + (-1)^{n-1}$ for all $n \geq 1$. Using the calculation from above, we get

$$\begin{aligned}
f(456, 789) &= f(2, 456)f(2, 789) \\
&= \left(2^{455} + (-1)^{455}\right)\left(2^{788} + (-1)^{788}\right) \\
&= \left(2^{455} - 1\right)\left(2^{788} + 1\right) \\
&= 2^{1243} - 2^{788} + 2^{455} - 1.
\end{aligned}$$

Proof that $f(m, n) = f(m, 2)f(2, n)$ for all positive integers m and n.

This proof is by strong induction and features calculations very similar to those in the solution to part (a). For notational convenience, we will refer to the equation $f(m, n) = f(m, 2)f(2, n)$ as Equation (2).

Since $f(2, 2) = 1$, we have that $f(2, n) = f(2, 2)f(2, n)$ and $f(m, 2) = f(m, 2)f(2, 2)$, which verifies Equation (2) in the case that $m = 2$ or $n = 2$. To see that Equation (2) holds when $m = 1$ for all n, first observe that $f(1, 1) = 2f(2, 1)$ and $f(1, 2) = 2f(2, 2)$. Now assume for some integer $r \geq 2$ that $f(1, n) = 2f(2, n)$ for $n = 1$, $n = 2$, and so on up to $n = r$. In particular, we assume $f(1,r) = 2f(2,r)$ and $f(1,r-1) = 2f(2,r-1)$. Then

$$\begin{aligned} f(1,r+1) &= f(1,r) + 2f(1,r-1) \\ &= 2f(2,r) + 2\left(2f(2,r-1)\right) \\ &= 2\left(f(2,r) + 2f(2,r-1)\right) \\ &= 2f(2,r+1) \end{aligned}$$

so $f(1,r+1) = 2f(2,r+1)$ as well. By strong induction, it follows that $f(1,n) = 2f(2,n)$ for all $n \geq 1$. Noting that $f(1,2) = 2$, this means $f(1,n) = f(1,2)f(2,n)$, so Equation (2) holds for all n when $m = 1$.

We will now use induction again to continue to show row-by-row that Equation (2) holds for all positive integers m and n. We know that it is true for all n when $m = 1$ and when $m = 2$. Suppose that $r \geq 2$ is an integer and that Equation (2) holds in the first r rows. In particular, we are assuming $f(r,n) = f(r,2)f(2,n)$ and $f(r-1,n) = f(r-1,2)f(2,n)$ for all $n \geq 1$. For any $n \geq 1$, we have

$$\begin{aligned} f(r+1,n) &= f(r,n) + 2f(r-1,n) \\ &= \left(f(r,2)f(2,n)\right) + 2\left(f(r-1,2)f(2,n)\right) \\ &= \left(f(r,2) + 2f(r-1,2)\right) f(2,n) \\ &= f(r+1,2)f(2,n) \end{aligned}$$

where the first and last equalities are by the definition of the entries in the $(r+1)^{\text{st}}$ row. By strong induction, it follows that $f(m,n) = f(m,2)f(2,n)$ for all positive integers m and n.

Can you see how to use the fact that $f(n,2) = f(2,n)$ for all positive integers n to prove that $f(m,n) = f(n,m)$ for all positive integers m and n?