# Problem of the Month

## Solution to Problem 3: December 2020

(a) In the first game, every arrangement is winnable. We will show this by describing how to win any arrangement.

Observe that every cell is a member of either one, two, or four $2 \times 2$ subgrids depending on if the cell is on an edge, in a corner, or in the "interior" of the grid. What is important for this argument is that every cell is in at least one $2 \times 2$ subgrid.

Suppose a coin is showing a tail and that it is the top-left coin in some $2 \times 2$ subgrid:

| | |
|---|---|
| $T$ | ? |
| ? | ? |

The question marks indicate that the coin in that cell could be showing either a head or a tail.

It is possible to change the indicated tail to a head without changing the other three coins. This can be done by performing the three moves below where the cells in which coins will be flipped are marked by an X.

| | |
|---|---|
| X | |
| X | X |

| | |
|---|---|
| X | X |
| X | |

| | |
|---|---|
| X | X |
| | X |

After performing these three legal moves, the coin in the top left corner has been flipped three times, so it will now be showing a head. The other three coins in the $2 \times 2$ subgrid were flipped twice each, so they will each be showing what they were showing before the moves were performed. No other coins in the grid were flipped, so this sequence of three legal moves has the effect of changing one coin from showing a tail to showing a head and does not change what any other coins are showing.

A similar sequence of three moves can be used to change a tail to a head if it is in one of the other corners of a $2 \times 2$ subgrid. Thus, it is possible to change any one tail to a head without changing any other coins, so every arrangement can be won by changing the tails to heads one at a time. We note that the technique for winning described above may not win in the smallest possible number of moves.

(b) A simple yet useful observation for this game is that the *parity* of the number of tails in any given row or column will not change as a result of a legal move.

Each legal move affects two rows and two columns and flips two coins in each of these two rows and two columns. For the cells of a row affected by a legal move, one of the following four situations occurs:

$$\begin{array}{|c|c|}\hline H & H \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline T & T \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline T & T \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline H & H \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline H & T \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline T & H \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline T & H \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline H & T \\ \hline\end{array}$$

In the two situations illustrated on the top, the number of tails in that row will either decrease by two or increase by two. In the two situations illustrated on the bottom, the number of tails in the affected row does not change. A similar argument shows that the number of tails in a column either increases by two, decreases by two, or stays the same after any legal move.

This means that if some row or column has an odd number of tails in the initial arrangement, then it will still have an odd number of tails after any number of legal moves. For an arrangement to be winnable, it must be possible to perform a sequence of legal moves resulting in the number of coins showing tails in each row and column to be 0. Since 0 is even, this means a winnable arrangement must have an even number of tails in each row and each column. We will call an arrangement of the coins "good" if there is an even number of tails in each row and an even number of tails in each column.

We just argued that if an arrangement is winnable, then it is good. We will next show that if an arrangement is good, then it is winnable. This will show that the winnable arrangements are exactly the good arrangements, so we will be able to count the winnable arrangements by counting the good arrangements.

Suppose an arrangement of the coins is good. A fundamental fact, which we argued above, is that a legal move transforms a good arrangement into an other good arrangement.

One strategy of winning is to systematically change all coins to show heads, row by row, starting with the top row. If there is a tail in the top row, locate the leftmost tail in the top row. Since the arrangement is good, the number of tails in the top row is even, so there must be at least one tail in the top row to the right of this tail. That is, the leftmost tail in the top row is in one of the first five columns. In the example illustrated below, it occurs in the third column. This means it is a legal move to flip the four coins with this tail in the top-left corner. Doing so will change the leftmost tail to a head.

$$\begin{array}{|c|c|c|c|c|c|}\hline H & H & T & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|}\hline H & H & H & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? \\ \hline ? & ? & ? & ? & ? & ? \\ \hline\end{array}$$

This move may change some heads to tails in the second row and may even change a head to a tail in the top row (if there was a head immediately to the right of the leftmost tail). However, as mentioned above, the new arrangement is still good, and the leftmost tail in the top row must now be further to the right.

After no more than four legal moves, the top row will either contain no tails or the leftmost tail will be in the fifth column. In the latter situation, the rightmost coin in the top row must also be showing a tail because the number of tails in the top row is even. Flipping the four coins in the top right corner of the grid will now make every coin in the top row show a head.

Once the top row has no tails in it, this procedure can be applied to the second row. Once the second row contains to tails, it can be applied to the third row, then the fourth, and finally the fifth. This means that a sequence of legal moves can be performed to leave the coins arranged so that every coin in the first five rows is showing a head.

However, this procedure cannot be applied to the bottom row since there is no "room". Flipping a coin in the bottom row necessarily flips coins in the fifth row, which has already been changed to show only heads. However, it turns out that since the arrangement was good, changing the coins in the first five rows to all show heads will force the coins in the sixth row to all show heads as well, meaning the game has already been won.

To see that this is true, suppose a good arrangement has no coins showing tails in the first five rows. In any column, if the coin in the bottom row showed a tail, then there would be an odd number of tails in that column. The arrangement is good, so this is not possible. Therefore, the coins in the bottom row must also be showing heads.

We have shown that every good arrangement is winnable, which shows that the good arrangements are exactly the winnable arrangements. To answer the question, we will count the good arranagements.

To do this, denote by $x$ the number of subsets of a set of 6 objects that have an even number of elements. With this notation, in any given row (or column), there are $x$ ways to arrange the coins so that an even number of them are showing tails. We will show that the number of good arrangements is $x^5$ and compute the exact value of $x$ later.

Notice that there are $x^5$ ways to arrange the coins in the first 5 rows so that there is an even number of tails in each of these rows. We will now prove the following claim: Every arrangement of an even number of tails in the first 5 rows can be "extended" in a unique way to a good arrangement.

To prove the claim, suppose the coins in the first 5 rows are arranged so that they each contain an even number of tails. To extend this to a good arrangement, we must decide how to arrange the coins in the bottom row. In order to have an even number of tails in each column, there is no choice to make: if there are an even number of tails in the first 5 cells of a column, then the final cell in that column must show a head. If there are an odd number of tails in the first 5 cells of a column, then the final coin must show a tail. Thus, if the arrangement of the coins in the first 5 rows can be extended to a good arrangement, then there is only one way to do it. All that remains is to show that the way of arranging the coins in the bottom row described above will produce an even number of tails in the bottom row.

This can be seen by observing that since there are an even number of tails in each column, there must be an even number of tails *in total*. Since the number of tails in the first 5 rows is even, it must be the case that the number of tails in the last row is even as well. Otherwise, the total number of tails in the grid would be odd.

You may wish to pause to dwell on the logic, but we have now shown that the number of good arrangements is equal to the number of ways to arrange the coins so that there are an even number of tails in each of the first five rows. As mentioned above, this leads to the number of good arrangements being equal to $x^5$. It remains to compute $x$.

Suppose $X$ is a set of 6 objects. We will count the number of subsets that have an even number of elements. That is, we will count the number of subsets of $X$ that contain exactly 0, 2, 4, or 6 elements.

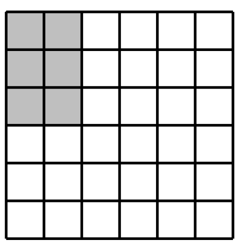
The empty set is the only subset of $X$ that contains 0 elements. Likewise, the set $X$ itself is the only subset of $X$ that contains 6 elements. There are $\dfrac{6 \times 5}{2} = 15$ subsets with 2 elements. This is because there are 6 ways to choose one element, 5 ways to choose a second, and this counts each subset exactly twice. "Choosing" 2 elements is the same as "ignoring" 4 elements, so the number of subsets with 4 elements is also equal to 15. Therefore, we have that $x = 1 + 15 + 15 + 1 = 32$.

In fact, it is not a coincidence that $x = 2^5$ and the total number of subsets of $X$ is $2^6$. Indeed, if $X$ is a set of $n$ elements for any any positive integer $n$, then there are $2^n$ subsets of $X$ in total, exactly $2^{n-1}$ of which contain an even number of elements. Put another way, exactly half of the subsets of a set have an even number of elements. You might want to try to prove this in general.

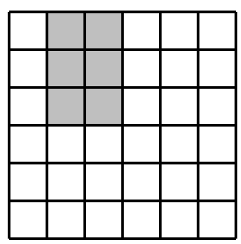
We can now answer the question: There are $(2^5)^5 = 2^{25}$ winnable arrangements.

(c) Throughout this solution, a sequence of legal moves will be called a "winning sequence" if it causes all coins in a given arrangement to show heads.

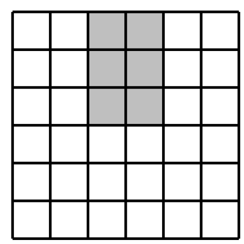
There are a total of 40 legal moves, but we will give names to 9 of them, pictured below.



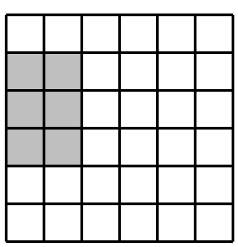
Move 1



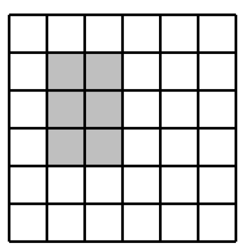
Move 2



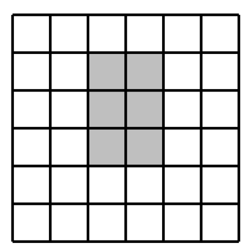
Move 3



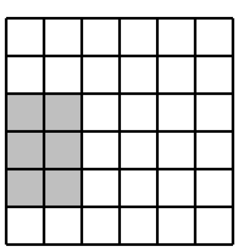
Move 4



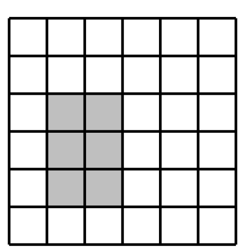
Move 5



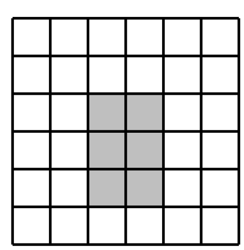
Move 6



Move 7



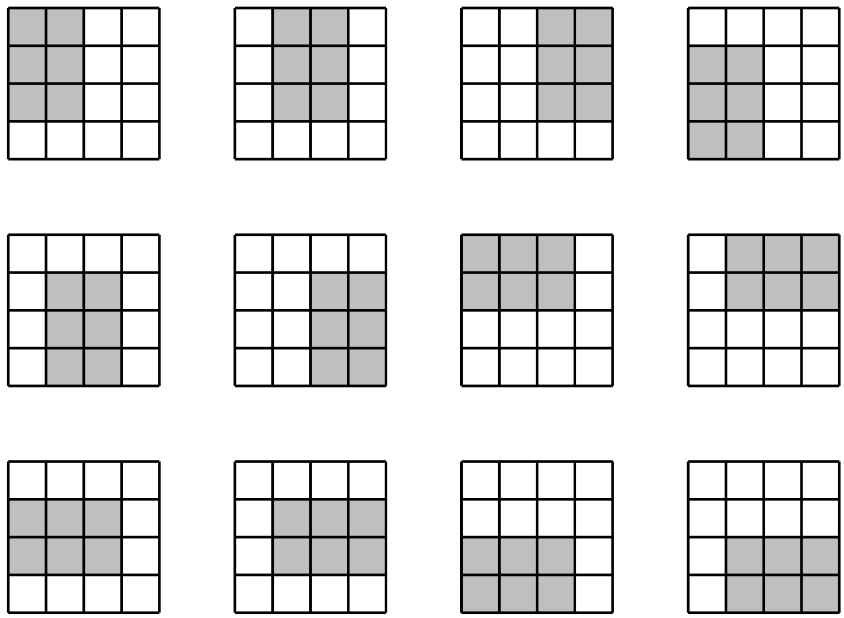
Move 8



Move 9

In the first part of this solution, we will argue that any winnable arrangement has a winning sequence that does not use any of Moves 1 through 9. This will allow us to show that there are at most $2^{40-9} = 2^{31}$ winnable arrangements. After that, we will demonstrate that there are at least $2^{31}$ winnable arrangements. These two facts together will imply that there are exactly $2^{31}$ winnable arrangements.

Consider any $4 \times 4$ subgrid. There are 12 moves that flip only coins in this subgrid:



Suppose we perform each of the moves above once, except the first one. The grid below indicates how many times each coin will be flipped. The highlighted cells are those in which the coin is flipped an odd number of times.

| | | | |
|---|---|---|---|
| 1 | 3 | 4 | 2 |
| 3 | 7 | 8 | 4 |
| 3 | 7 | 8 | 4 |
| 2 | 4 | 4 | 2 |

If a coin is flipped an odd number of times, it has changed from showing a head to showing a tail, or vice versa. If a coin is flipped an even number of times, there will be no change in what it shows. This means the effect of the first move can be "simulated" by the other eleven. That is, if a move flips coins in the top-left $3 \times 2$ subgrid of a $4 \times 4$ subgrid, its effect can be achieved using the other eleven moves that flip coins only in that $4 \times 4$ subgrid. This is what will allow us to "eliminate" Moves 1 through 9.

Suppose we have a winnable arrangement and a winning sequence. By replacing each (if any) occurrence of Move 1 by the eleven other moves described above, we get a new winning sequence that does not use Move 1. This new sequence will be longer than the original and will have new occurrences of Moves 2, 3, 4, 5, and 6. The key is that if an arrangement is winnable, then there is a winning sequence that does not use Move 1.

Next, consider Move 2. The eleven moves that can be used to replace an occurrence of Move 2 include Moves 3, 5, and 6 (among others), but do not include Move 1. We can now take our winning sequence (that does not include Move 1) and replace every occurrence of Move 2 by eleven other moves. Again, the sequence will get longer and there will be new occurrences of Moves 3, 5, and 6, but Move 1 will not be reintroduced. Thus, if an arrangement is winnable, then there is a winning sequence that does not use Move 1 and does not use Move 2.

Next, each occurrence of Move 3 can be replaced by eleven other moves. Considering the diagrams above carefully, this will not reintroduce Move 1 or Move 2 to the sequence. We can then continue this way to see that if there is a winning sequence, then there is a winning sequence that does not use any of Moves 1 through 9. The new sequence will have more moves, but the key is that it uses at most $40 - 9 = 31$ distinct moves. [It is not directly important for this argument, but removing nine moves is in fact the "best" that we can do. By the end of the solution, you might have a better idea of what this means and why it is true.]

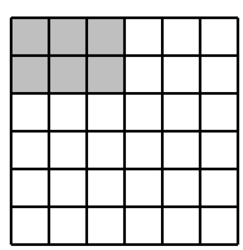
We can further refine the winning sequence by removing repetitions. As noted earlier and mentioned in the hint, the order in which moves are applied does not matter. This is because the overall effect of a sequence of moves on an individual coin is only influenced by how many times that coin is flipped, not by when it is flipped. Thus, to know what effect a sequence has on a given coin, one needs only to count how many times it is flipped. Performing any individual move twice contributes an even number of flips of each coin (either 0 or 2), which means that we can get a shorter winning sequence by eliminating "pairs" of the same move.

Therefore, if an arrangement is winnable, then there is a winning sequence that uses at most 31 moves and does not use any move more than once. Since the order does not matter, there are essentially only $2^{31}$ winning sequences: For each of the 31 moves, we either use it or do not use it. Since every winning arrangement can be won in one of $2^{31}$ ways, there cannot be more than $2^{31}$ winnable arrangements.

We will show that there are at least $2^{31}$ winnable arrangements by building as many. This will be done using reasoning similar to that in part (b). It should be pointed out that in this part of the argument, we are only concerned with whether or not an arrangement is winnable, so we will not worry about whether or not we are using Moves 1 through 9.

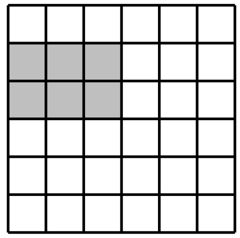
Suppose three consecutive coins in a row are flipped. If the number of tails in that row was even, it will now be odd, and if the number of tails in that row was odd, it will now be even. This can be seen using a similar argument to the one at the beginning of the solution to part (b).

Following similar reasoning to part (b), we can change the coins in the grid to heads row by row. Starting with the first row, if there are an odd number of tails, apply the move indicated below:



This will flip three coins in the first row making the number of tails in the first row even. By using the five moves that flip exactly two coins in the top row (the "$3 \times 2$" moves), the top row can be changed so that it shows all heads using the same reasoning as in part (b). This will potentially change some heads to tails *below* the top row, but the important thing is that the top row will contain all heads.

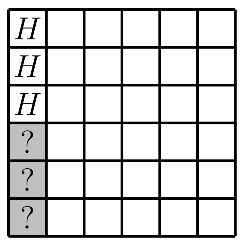
Next, the same idea can be used to change all coins in the second row to show heads. If there are an odd number of tails in this row, apply the move



This will change the coins so that there are an even number of tails in the second row. Using the five moves that flip exactly two coins in the second row and no coins in the top row, the second row can be changed so that all coins show a head. In a similar way, the coins can be changed so that there are no tails in the top three rows and all remaining tails are in the bottom three rows.

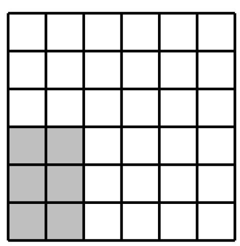
We will now argue that it is possible to perform a sequence of moves to get all remaining tails in the $3 \times 2$ subgrid in the bottom-right corner of the array.

First, focus on the three cells highlighted below:



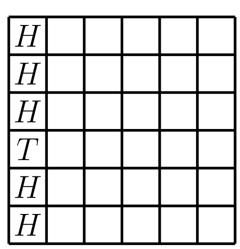

If the coins in these cells all show heads, then there is nothing to do. Otherwise, there are seven possibilities.

If all three of them show a tail, then the move

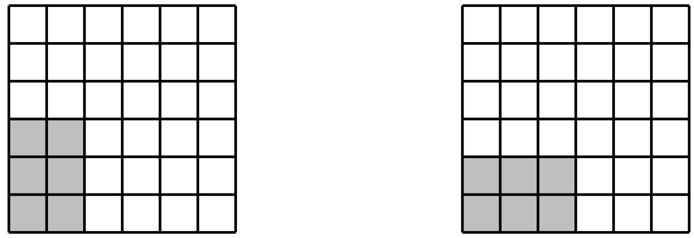


will not change any coins in the top three rows, and will make all coins in the first (leftmost) column show heads.

For each of the other six possibilities, it is possible to change the coins in the first column to all show heads without changing any coins in the first three rows. For example, if the coins in the first column are arranged




then the two moves below will achieve this:



The other five situations can be dealt with similarly, in each case using at most three moves.

This argument can be repeated to ensure that the first four columns contain all heads. Therefore, it is possible, no matter how the coins were arranged, to use a sequence of legal moves to convert the arrangement to

| | | | | | |
|---|---|---|---|---|---|
| $H$ | $H$ | $H$ | $H$ | $H$ | $H$ |
| $H$ | $H$ | $H$ | $H$ | $H$ | $H$ |
| $H$ | $H$ | $H$ | $H$ | $H$ | $H$ |
| $H$ | $H$ | $H$ | $H$ | ? | ? |
| $H$ | $H$ | $H$ | $H$ | ? | ? |
| $H$ | $H$ | $H$ | $H$ | ? | ? |

so that all remaining tails (if any) are in the cells marked with question marks.

We will now explain how to construct $2^{31}$ winnable arrangements. For brevity, we will refer to the coins in the 30 cells outside those marked with question marks above as the "first 30" and the other 6 coins as the "last 6". We will show that for every arrangement of the first 30 coins, there are at least two arrangements of the last 6 coins that make the overall arrangement winnable.

Consider an arbitrary arrangement of the first 30 coins and arrange the last 6 coins to all show heads. We do not claim that this arrangement is winnable, but it will help us to find winnable arrangements.

In the way described earlier, perform a sequence of legal moves so that all remaining tails (if any) are among the last 6 coins. For example, perhaps the last 6 coins are arranged as indicated below:

| | |
|---|---|
| $H$ | $H$ |
| $T$ | $T$ |
| $H$ | $H$ |

The overall effect of this sequence of legal moves is to do two things: change all of the first 30 coins to show heads, and flip the coins in the last 6 that are now showing tails. In the example above, it flips the two coins in the middle. To construct a winnable arrangement, arrange the first 30 coins in the same way as before but arrange the last 6 exactly as they are shown above. The sequence of legal moves will still make the first 30 coins show heads, but since we know the sequence flips the "middle two" of the last 6 coins, the sequence will actually change the whole grid to show heads.

In general, we can generate a winnable arrangement by following these steps.

(i) Arrange the first 30 coins arbitrarily and the last 6 to show heads.

(ii) Perform a sequence of legal moves so that all tails (if any) are among the last 6 coins.

(iii) A winnable arrangement can now be found by arranging the first 30 coins in the same way as in (i), but arranging the last 6 in the way they appeared *after* (ii).

The sequence of moves in (ii) will change the arrangement in (iii) to show all heads. This is because in (ii) we learn the overall effect of the moves on the last 6 coins, so in (iii) we can set them up so that they will be flipped to show heads. Thus, we get at least one winnable arrangement corresponding to the given arrangement of the first 30 coins.

To get another, in step (iii) instead arrange the last 6 coins in a way opposite to how they appeared after step (ii). This is a different arrangement because all of the last 6 coins will be different from the other arrangement. Performing the sequence of legal moves will now result in the last 6 coins all showing tails. The arrangement can then be changed to show all heads by flipping all of the last 6 coins, which is a legal move.

Therefore, there are at least $2 \times 2^{30} = 2^{31}$ winnable arrangements. In fact, this also explains how to win. That is, perform a sequence of legal moves to force all tails to the last 6 coins. If the arrangement was winnable, then these coins will either all show tails or all show heads. At most one more move will convert all coins to heads. If any other arrangement appears in the last 6 coins, then the arrangement was not winnable in the first place.

It is interesting to note that things actually do not get much more complicated if the games are played on an $n \times n$ grid with $n > 6$. In part (a), the same argument shows that every arrangement is winnable. In part (b), if the game were played on an $n \times n$ grid, then $2^{(n-1)\times(n-1)}$ of the $2^{n \times n}$ arrangements are winnable. Notice that this means that if the coins are arranged randomly, then the probability that the arrangement is winnable is

$$\frac{2^{n^2-2n+1}}{2^{n^2}} = \frac{1}{2^{2n-1}}$$

which gets very small as $n$ gets large. Thus, if your friend arranges the coins randomly in the game in (b), there is a very small chance that the arrangement is winnable.

On the other hand, extending the reasoning in part (c) shows that there are $2^{n^2-5}$ winnable arrangements, so the probability that a random arrangement is winnable in the game in (c) is

$$\frac{2^{n^2-5}}{2^{n^2}} = \frac{1}{2^5} = \frac{1}{32}$$

which does not depend on $n$.