Problem of the Month

Problem 4: January 2021

Problem

In this problem, we will explore when a quadratic polynomial of the form $x^2 + ux + v$ can be decomposed as the sum of the squares of two other polynomials. Keep in mind that a constant function is a polynomial. All polynomials in the problem statements below are assumed to have real coefficients, though they may not have real roots.

(a) Find at least three pairs $\big(p(x), q(x)\big)$ of polynomials such that $(p(x))^2 + (q(x))^2 = x^2 + 2x + 2$.

(b) Suppose $f(x) = x^2 + ux + v$ has the property that $f(x) \geq 0$ for all real numbers x. Prove that there are polynomials $p(x)$ and $q(x)$ such that $x^2 + ux + v = (p(x))^2 + (q(x))^2$.

In the remaining parts of this problem, we will say that the pair of polynomials $\big(p(x), q(x)\big)$ is *special* for the polynomial $x^2 + ux + v$ if

- the coefficients of $p(x)$ and $q(x)$ are all rational, and
- $x^2 + ux + v = (p(x))^2 + (q(x))^2$.

(c) Prove that there are no special pairs for $x^2 + x + 1$.

(d) Prove that if there is a special pair for $x^2 + ux + v$, then u and v are both rational and there is a rational number r such that $4v - u^2 = r^2$.

(e) Prove that if there is a special pair for $x^2 + ux + v$, then there are infinitely many special pairs for $x^2 + ux + v$.

Hint

(a) Expand the expression on the left side of $(ax+b)^2+(cx+d)^2 = x^2+2x+2$ and look at the resulting coefficients.

(b) What if $q(x)$ is constant?

(c) If real numbers s and t satisfy $s^2+t^2=1$, then there is a real number θ with the property that $s=\cos\theta$ and $t=\sin\theta$.

(d) Compute $4v-u^2$ in terms of the coefficients of $p(x)$ and $q(x)$.

(e) Try to link Pythagorean triples to angles θ with the property that $\cos\theta$ and $\sin\theta$ are both rational.