Problem of the Month

Problem 5: February 2021

Problem

For an integer $n \geq 3$, we define T_n to be the triangle with side lengths $n-1$, n, and $n+1$, and define A_n to be the area of T_n. We will say that an integer $n \geq 3$ is *remarkable* if A_n is an integer.

(a) Determine all integers n for which T_n is right-angled.

(b) Suppose n is a remarkable integer. Prove that

  (i) $\dfrac{n^2-4}{3}$ is a perfect square,

  (ii) n is not a multiple of 3, and

  (iii) n is even.

(c) There are three remarkable integers less than or equal to 100. Determine these three integers.

(d) The only remarkable integers between 100 and 10 000 are $n = 194$, $n = 724$, and $n = 2702$. Find a polynomial function $f(n)$ of degree greater than 1 with the property that if n is a remarkable integer, then $f(n)$ is also a remarkable integer. Use this polynomial to deduce that there are infinitely many remarkable integers.

(e) Explain how to find all remarkable integers. This should involve somehow describing an infinite set of remarkable integers as well as justification that your set is complete. Keep in mind that the infinite set from part (d) may not include *all* remarkable integers.

Hint

(a) No hint given.

(b) Using Heron's formula, it is possible to find an expression for A_n in terms of n.

(c) Sometimes it is faster to check all possibilities than to find a more clever approach. The conditions in part (b) can be used to eliminate about two thirds of the integers between 3 and 100.

(d) There is at least one such polynomial of degree 2.

(e) Try to find a few pairs (a, b) of positive integers that satisfy the equation $a^2 - 3b^2 = 1$. Compare the values of a to the known remarkable integers. Factoring the equation above as $(a + b\sqrt{3})(a - b\sqrt{3}) = 1$ and taking small positive integer powers of both sides may provide some insight into how one might generate more integer solutions to $a^2 - 3b^2 = 1$.