



Problem of the Month

Problem 7: April 2021

Problem

For an integer $n \geq 3$, n^2 points form an $n \times n$ square grid.

Define $P(n)$ to be the probability that three distinct points randomly selected from the grid are the vertices of a triangle with positive area. Also define $f(n)$ to be the number of sets of three distinct points from the grid that lie on a common line. We can think of $f(n)$ as the number of sets of three distinct points from the grid that are the vertices of a triangle with area 0.

For instance, with $n = 3$, it can be shown that there are 84 possible ways to select three distinct points, that 8 of the sets of three points lie on a line, and that 76 of the sets of three points form the vertices of a triangle with positive area. Thus, $f(3) = 8$ and $P(3) = \frac{76}{84} = \frac{19}{21}$.

The goal of this problem is to estimate $P(n)$ for large n . The approach outlined will be to estimate $f(n)$ and use it to estimate $P(n)$.

- When $n = 3$, $f(n) = 8$ and $P(3) = \frac{19}{21}$. Compute $f(n)$ and $P(n)$ for $n = 4$ and $n = 5$.
- For $n \geq 3$, prove that $f(n+1) < f(n) + 5n^4 + 5n^3 + 5n^2 + 5n$. This will allow us to understand how quickly $f(n)$ grows which will help to estimate $P(n)$.
- Using part (b), prove that $f(n) < n^5$ for all $n \geq 3$.
- Prove that there is a constant c with the property that $P(n) > 1 - \frac{c}{n}$ for all $n \geq 3$. Use this to explain why the following statement makes sense: “For very large n , it is nearly certain that three points selected randomly from an $n \times n$ grid will be the vertices of a triangle with positive area.”

As indicated in part (d), this problem is meant to examine what happens to $P(n)$ as n gets large. Since it seems very difficult to calculate $f(n)$ (and hence, $P(n)$) directly for large n , we instead estimate its value. As long as we carefully keep track of how good/bad the estimates can be, we can say something meaningful about $P(n)$ for large n without actually computing it directly. Very frequently, mathematicians use estimates like these when exact answers are difficult or impossible to obtain. These estimates are often as useful as exact answers.



Hint

- (a) Drawing a picture (or a few pictures) could be helpful. Most of the work is in computing $f(n)$. One way to do this is to consider the various slopes that a line through three points in the grid can have. It might also be useful to do an internet search on *binomial coefficients*. A binomial coefficient, sometimes denoted $\binom{n}{k}$ (read “ n choose k ”), is equal to the number of ways to choose k objects from a set of n distinct objects. It can be computed as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
- (b) Consider breaking an $(n+1) \times (n+1)$ grid into two sets of points: the bottom-left $n \times n$ subgrid and the $2n+1$ points along the top and right. If three points lie on a line, how can they be distributed among these two sets? Remember, the goal is not to compute $f(n+1)$ exactly, but to bound it by $f(n) + 5n^4 + 5n^3 + 5n^2 + 5n$. In fact, you might be able to do better than this by showing, for example, that $f(n+1) < f(n) + 2n^4 + 2n^3 + 2n^2 + 2n$, or some other bound involving $f(n)$.
- (c) Expand $(n+1)^5$.
- (d) Use part (c). There are many values of c that will work. It will likely be helpful to show some other inequalities like $n^2 - 1 \leq \frac{1}{2}n^2$ for $n \geq 3$.
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