



## Problem of the Month

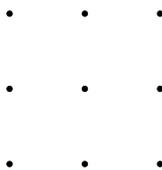
### Solution to Problem 7: April 2021

As suggested in the hint, will use *binomial coefficients* throughout this solution. The symbol  $\binom{m}{k}$  (in words, “ $m$  choose  $k$ ”) represents the number of ways to choose  $k$  objects from  $m$  distinct objects. We will use explicitly that  $\binom{m}{2} = \frac{m(m-1)}{2}$  and  $\binom{m}{3} = \frac{m(m-1)(m-2)}{6}$ , but you may wish to explore this standard notation more generally.

We will also use the standard terminology that three points are *collinear* if they are on a common line.

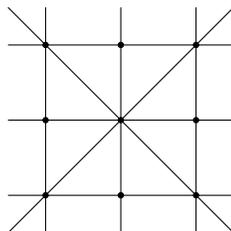
- (a) Although it was not part of the problem, we will show that  $f(3) = 76$  and  $P(3) = \frac{19}{21}$ .

Consider the  $3 \times 3$  grid below:



We can draw three vertical lines and three horizontal lines such that each line passes through exactly three points. This accounts for a total of 6 ways to choose three distinct collinear points from the grid.

If three points are collinear but the line they define is neither vertical nor horizontal, then each point must be in a different row and a different column. The only sets of three collinear points with one point in each row and each column are the two “diagonals”. This gives two more sets of three points, for a total of 8. Thus,  $f(3) = 8$ . The diagram below shows all eight of the lines that contain three points:



To compute  $P(3)$ , we will need the total number of ways to choose three distinct points from the  $3 \times 3 = 9$  points in the grid. This is equal to  $\binom{9}{3} = \frac{9 \times 8 \times 7}{6} = 84$ .

There are 84 ways to choose three distinct points, and there are  $f(3) = 8$  ways to choose three distinct points that are collinear. Each of the remaining  $84 - 8 = 76$  sets of three distinct points are the vertices of a triangle of positive area, so  $P(3) = \frac{76}{84} = \frac{19}{21}$ .

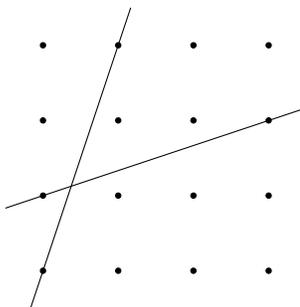
To compute  $P(4)$ , we will compute  $f(4)$  and use that the number of ways to choose three distinct points from the  $4 \times 4 = 16$  points in the grid is  $\binom{16}{3} = 560$ . Then we can compute

$$P(4) = \frac{560 - f(4)}{560}.$$

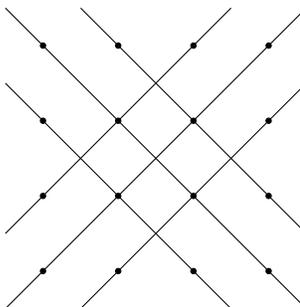
To count the sets of three collinear points, we will first find all lines that pass through at least two of the points. To make sure we do not miss any, we will examine the possible slopes of lines through at least two points, imagining that the bottom-left point is at the origin and the others are the points  $(a, b)$  where  $0 \leq a \leq 3$  and  $0 \leq b \leq 3$ . If two points are chosen, then the slope can be computed as  $\frac{\text{“rise”}}{\text{“run”}}$ . Ignoring vertical and horizontal lines for now, the possible rises are  $-3, -2, -1, 1, 2,$  and  $3$  and the possible runs are the same set of values. Thus, the possible slopes of lines that are neither vertical nor horizontal are

$$\pm 3, \pm 2, \pm \frac{3}{2}, \pm 1, \pm \frac{2}{3}, \pm \frac{1}{2}, \pm \frac{1}{3}$$

If a line that is neither horizontal nor vertical passes through three distinct points, then these three points must be in three different rows and three different columns. Suppose such a line has slope 3. Then the two points on this line that are farthest from each other must be a vertical distance of at least 2 apart, and hence, must be a vertical distance of at least 6 apart. There are only four rows of points, so it is impossible for a line of slope 3 to pass through three points in the grid. Similarly, a line of slope  $\frac{1}{3}$  cannot pass through three points in the grid. The diagram below may help to illustrate this.



Similar arguments can be used to show that a line of any of the slopes above, except 1 and  $-1$ , can pass through at most two points in the grid. Thus, if a line passes through three or more points in the grid, it must have slope  $\pm 1$  or be horizontal or vertical. There are six lines of slope  $\pm 1$  that pass through at least three points: Two diagonals, one above each diagonal, and one below each diagonal. They are shown below:



The four lines other than the diagonals each pass through exactly three points in the grid. Thus, we get four sets of three collinear points.

Each diagonal passes through four points in the grid. From each such line, we can choose a set of three collinear points by ignoring one of the four points. Thus, each of these two lines contributes another four sets of three collinear points. So far, we have counted  $4 + 4 + 4 = 12$  sets of three collinear points in the grid.

Each horizontal line passes through four points. By the same reasoning as in the previous paragraph, the horizontal lines each contribute four sets of three collinear points for a total of  $4 \times 4 = 16$  more sets of collinear points. We similarly get 16 sets of three collinear points from the four vertical lines. In total,  $f(4) = 12 + 16 + 16 = 44$ . We can now compute  $P(4)$  as

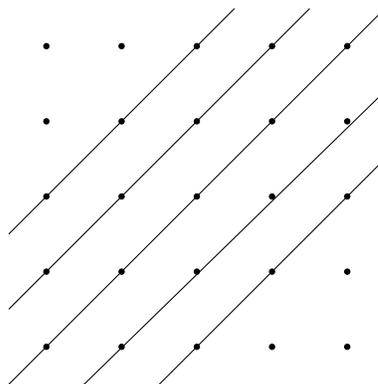
$$P(4) = \frac{560 - f(4)}{560} = \frac{560 - 44}{560} = \frac{516}{560} = \frac{129}{140}.$$

To compute  $f(5)$ , we will again examine the possible slopes of lines through at least two points in the grid. In a  $5 \times 5$  grid, the possible rises of a line that is neither vertical nor horizontal are 1, 2, 3, and 4, and the possible runs are the same. The possible slopes coming from these rises and runs are

$$\pm 4, \pm 3, \pm 2, \pm \frac{3}{2}, \pm \frac{4}{3}, \pm 1, \frac{3}{4}, \pm \frac{2}{3}, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}$$

Using reasoning similar to the case for  $n = 4$ , it can be shown that of the slopes listed above, only a line with slope  $\pm 1$ ,  $\pm \frac{1}{2}$ , or  $\pm 2$  can pass through three points in a  $5 \times 5$  grid. By examining the lines having these slopes, as well as the vertical and horizontal lines, we can compute  $f(5)$ .

There are five lines of slope 1 that pass through at least three points: a diagonal, two lines above it, and two lines below it. These five lines are depicted below.

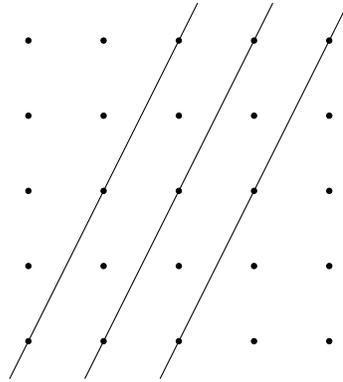


Of these five lines, there are two that pass through exactly three points in the grid. Thus, each of these two lines contributes one set of three collinear points. As argued in the  $n = 4$  case, the two lines through four points each contribute four sets of three collinear points. The one line through five points contributes  $\binom{5}{3} = 10$  sets of three collinear points. Thus, from the lines of slope 1, we get a total of  $1 + 1 + 4 + 4 + 10 = 20$  sets of three collinear points. Similar reasoning can be used to show that there are 20 sets of three collinear points on the lines of slope  $-1$ .

The counts for each of the slopes  $\pm \frac{1}{2}$  and  $\pm 2$  are essentially the same, so we will only explicitly examine the case when the slope is 2.

Suppose points  $A$ ,  $B$ , and  $C$  are three collinear points in the grid so that the slope of the line through the points is 2. Also suppose that  $A$  and  $C$  are the two points that are farthest apart. Since the line is neither horizontal nor vertical,  $A$ ,  $B$ , and  $C$  are in different columns and so the horizontal distance between  $A$  and  $C$  is at least 2. Since the slope is 2, this means the vertical distance between  $A$  and  $C$  is at least 4. However, 4 is the largest possible vertical distance between two points in the grid, so this means the vertical

distance between  $A$  and  $C$  is exactly 4. Therefore, one of  $A$  and  $C$  must be in the bottom row. There are only five lines of slope 2 passing through one of the points in the bottom row, and only three of them pass through at least three points in the grid. In fact, each of these three lines passes through exactly three points in the grid, and they are depicted below:



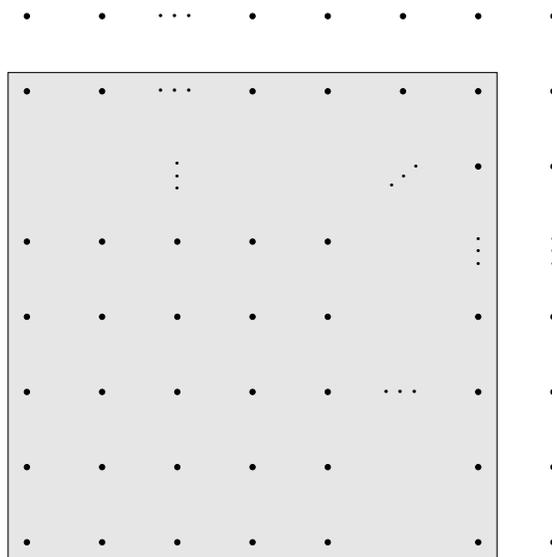
This means that we get an additional three sets of three collinear points from the lines of slope 2. Similarly, we get three sets from the lines of slopes  $-2$ ,  $\frac{1}{2}$ , and  $-\frac{1}{2}$ .

Each vertical line and each horizontal line contains another  $\binom{5}{3} = 10$  sets of three collinear points, so we get another  $10 \times 10 = 100$  sets of collinear points. Combining with the earlier counts, have that  $f(5) = 20 + 20 + 4 \times 3 + 100 = 152$ .

The total number of sets of three points in a  $5 \times 5$  grid is  $\binom{25}{3} = 2300$ , so

$$P(5) = \frac{2300 - 152}{2300} = \frac{2148}{2300} = \frac{537}{575}.$$

(b) We can consider an  $(n + 1) \times (n + 1)$  grid as an  $n \times n$  grid with  $2n + 1$  additional points:



We will denote by  $A$  the set of points in the highlighted  $n \times n$  grid (the bottom left  $n \times n$  grid) and by  $B$  the remaining  $2n + 1$  points. Within  $B$ , we will refer to the  $n + 1$  points in the rightmost column as the “vertical part” and the  $n + 1$  points in the top row as

the “horizontal part”. Note that the top right point is in both the vertical part and the horizontal part of  $B$ .

There are four possibilities for a set of three collinear points in the  $(n + 1) \times (n + 1)$  grid: They are all in  $A$ , two are in  $A$  and one is in  $B$ , one is in  $A$  and two are in  $B$ , or they are all in  $B$ .  $f(n + 1)$  is the sum of the number of sets of collinear points in each case. Remember that our goal is not to compute  $f(n + 1)$  precisely, but to show that it is smaller than  $f(n) + 5n^4 + 5n^3 + 5n^2 + 5n$ .

Case 1: All three points are in  $A$ . By the definition of  $f(n)$ , there are exactly  $f(n)$  sets of three collinear points in  $A$ .

Case 2: Two points are in  $A$  and one is in  $B$ . For any two points in  $A$ , the line defined by the two points intersects  $B$  at most once in the vertical part and at most once in the horizontal part. This means for any two distinct points in  $A$ , the line through them intersects  $B$  at most twice in total. Thus, the number of sets of three collinear points in this case is *no more than* two times the number of ways to choose two distinct points from  $A$ . There are  $n^2$  points in  $A$ , so there are  $\binom{n^2}{2} = \frac{n^2(n^2 - 1)}{2}$  pairs of distinct points in  $A$ . In this case, there are at most  $2 \times \frac{n^2(n^2 - 1)}{2} = n^4 - n^2$  sets of three distinct points.

Case 3: One point is in  $A$  and two points are in  $B$ . The line defined by two points in the horizontal part of  $B$  does not contain any points in  $A$ . Similarly, the line defined by any two points in the vertical part of  $B$  does not contain any points in  $A$ . Therefore, for a set of three collinear points to fall in this case, we must have one of the two points in  $B$  in the vertical part and one of the two points in the horizontal part. Neither of these two points can be the top-right point since this would make the line either vertical or horizontal. Therefore, there are  $n \times n$  possible ways to choose the two points from  $B$ . The line defined by these two points may contain no points from  $A$  or could contain several. Since we do not need to count precisely, it will be sufficient to observe that the line intersects each column at most once. Thus, for any of the  $n^2$  pairs of points from  $B$  described above, there are at most  $n$  points from  $A$  on that line. Therefore, there are at most  $n^3$  sets of three collinear points in this case.

Case 4: All three points are in  $B$ . Since there are three points, either two of the points are in the vertical part or two of the points are in the horizontal part. Two points define a line, so this means the line must be either horizontal or vertical. Thus, in fact, either all three points are in the vertical part or all three points are in the horizontal part.

There are  $n + 1$  points in the horizontal part and  $n + 1$  points in the vertical part (the top-right point is in both parts). Thus, there are  $2 \binom{n + 1}{3} = 2 \frac{(n + 1)(n)(n - 1)}{6} = \frac{n^3 - n}{3}$  sets of three collinear points in this case.

We have shown that there are  $f(n)$  sets of three collinear points in Case 1 and that there are  $\frac{n^3 - n}{3}$  sets of three collinear points in Case 4. As well, we showed that there are at most  $n^4 - n^2$  sets of three collinear points in Case 2 and that there are at most  $n^3$  sets of three collinear points in Case 3. In Cases 2 and 3, you might have noticed that we were not very careful about our counting. For instance, in Case 3, it is not difficult to show that the line defined by the two points in  $B$  intersects  $A$  at most  $n - 2$  times (a better bound than  $n$ ) and in fact, likely intersects  $A$  even fewer times. However, these rather simple

bounds will be sufficient.

Putting the information from Cases 1 through 4 together, we have

$$f(n+1) \leq f(n) + (n^4 - n^2) + n^3 + \frac{n^3 - n}{3} = f(n) + n^4 + \frac{4}{3}n^3 - n^2 - \frac{1}{3}n.$$

Since  $n$  is a positive integer,  $n^4 < 5n^4$ ,  $\frac{4}{3}n^3 < 5n^3$ ,  $-n^2 < 5n^2$ , and  $-\frac{1}{3}n < 5n$ . We can apply these four inequalities to get

$$f(n+1) < f(n) + 5n^4 + 5n^3 + 5n^2 + 5n.$$

**Note:** The last step in the solution to (b) might seem like a strange thing to do since we already had a “better” bound on  $f(n+1)$ . It might seem like we threw away information, and in fact, we did. Roughly speaking, we have sacrificed some accuracy in order to get an expression that will be easier to work with. In general, when mathematicians use this kind of technique, it can be quite delicate trying to balance how much accuracy can be sacrificed while making the quantities and expressions involved easy enough to manage. It often involves going back to earlier parts of a solution several times to make an adjustment. Indeed, when we were writing this problem, part (c) was the last to be finalized. This is because, after writing a solution to part (d), we went back to adjust what we asked for in part (c). You may be able to solve part (d) by using different versions of parts (b) and (c).

- (c) Continuing with our estimation in part (b), we have for  $n \geq 3$  that  $5n^2 \leq 10n^2$  and  $5n^3 \leq 10n^3$ . Thus, for all integers  $n \geq 3$ , we actually have that

$$\begin{aligned} f(n+1) &< f(n) + 5n^4 + 10n^3 + 10n^2 + 5n \\ &< f(n) + 5n^4 + 10n^3 + 10n^2 + 5n + 1 \end{aligned}$$

where the addition of 1 at the end will be used shortly.

Using the calculations in part (a), we get that  $f(3) = 8 < 243 = 3^5$ ,  $f(4) = 44 < 1024 = 4^5$ , and  $f(5) = 152 < 3125 = 5^5$ , so  $f(n) < n^5$  for each of  $n = 3$ ,  $n = 4$ , and  $n = 5$ . We will now use induction to prove that  $f(n) < n^5$  for all  $n \geq 3$ .

To do this, we will assume that  $f(k) < k^5$  for some integer  $k \geq 3$  and from this deduce that  $f(k+1) < (k+1)^5$ . Expanding  $(k+1)^5$ , we get  $(k+1)^5 = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1$ . Thus, using the inequality above and the assumption that  $f(k) < k^5$ , we have

$$\begin{aligned} f(k+1) &< f(k) + 5k^4 + 10k^3 + 10k^2 + 5k + 1 \\ &< k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 \\ &= (k+1)^5 \end{aligned}$$

Therefore, if the statement “ $f(k) < k^5$ ” is true for some integer, then it is true for the next integer. By the principle of mathematical induction,  $f(n) < n^5$  for all  $n \geq 3$ .

**Note:** It might be a little clearer now why we used such a “weak” bound in part (b). The calculation above was very easy because we bounded  $f(n+1)$  by  $f(n)$  plus part of the expression  $(n+1)^5$ .

- (d) In an  $n \times n$  grid, the number of ways to choose three points is  $\binom{n^2}{3} = \frac{n^2(n^2-1)(n^2-2)}{6}$ .

If we call this quantity  $g(n)$ , then

$$P(n) = \frac{g(n) - f(n)}{g(n)} = 1 - \frac{f(n)}{g(n)} = 1 - \frac{6f(n)}{n^2(n^2-1)(n^2-2)}.$$

We will find a constant  $c$  so that

$$\frac{6f(n)}{n^2(n^2-1)(n^2-2)} < \frac{c}{n}$$

and hence

$$-\frac{6f(n)}{n^2(n^2-1)(n^2-2)} > -\frac{c}{n}$$

which will mean that

$$P(n) = 1 - \frac{6f(n)}{n^2(n^2-1)(n^2-2)} > 1 - \frac{c}{n}$$

for  $n \geq 3$ .

From part (c), we have that  $f(n) < n^5$  for all  $n \geq 3$ , so we get

$$\frac{6f(n)}{n^2(n^2-1)(n^2-2)} < \frac{6n^5}{n^2(n^2-1)(n^2-2)} = \frac{6n^3}{(n^2-1)(n^2-2)}. \quad (*)$$

Now notice that for  $n \geq 3$ , we have  $n^2 - 1 > \frac{1}{2}n^2$ . To see this, observe that  $n^2 > 2$  when  $n \geq 3$ , so  $2n^2 - n^2 > 2$  which can be rearranged to  $2n^2 - 2 > n^2$ . Multiplying through by  $\frac{1}{2}$  gives the result.

In a similar way, it can be argued that  $n^2 - 2 > \frac{1}{2}n^2$  when  $n \geq 3$ .

Since  $n^2 - 1 > \frac{1}{2}n^2$  and  $n^2 - 2 > \frac{1}{2}n^2$ , we have  $\frac{1}{n^2-1} < \frac{2}{n^2}$  and  $\frac{1}{n^2-2} < \frac{2}{n^2}$ . Therefore,

$$\begin{aligned} \frac{6n^3}{(n^2-1)(n^2-2)} &= 6n^3 \times \frac{1}{n^2-1} \times \frac{1}{n^2-2} \\ &< 6n^3 \times \frac{2}{n^2} \times \frac{2}{n^2} \\ &= \frac{24}{n} \end{aligned}$$

Combining this with (\*), we have

$$\frac{6f(n)}{n^2(n^2-1)(n^2-2)} < \frac{24}{n}$$

for all  $n \geq 3$ . Thus,

$$P(n) = 1 - \frac{6f(n)}{n^2(n^2-1)(n^2-2)} > 1 - \frac{24}{n}.$$

This means we can take  $c = 24$ . In fact, any  $c$  larger than 24 will work.

Finally, we discuss what this means for large  $n$ . Suppose, for example, that  $n = 24 \times 10^6$ . Then the above inequality implies

$$P(n) > 1 - \frac{24}{24 \times 10^6} = 0.999999.$$

So for this value of  $n$ , the probability that the three chosen points form a triangle of positive area is *at least* 0.999999, which means it is very close to 1. As  $n$  gets even larger, the quantity  $\frac{24}{n}$  gets even smaller, so  $P(n)$  is forced to be even closer to 1. While this inequality never tells us the exact probability, it does give us a useful estimate.

It is worth pointing out that with  $n = 3$ , this inequality tells us that  $P(n) > 1 - \frac{24}{3}$  so  $P(n) > -7$ . This is true, but it isn't very interesting since probabilities are always positive. This is typical of the type of argument that we have used. The result – that  $P(n)$  is very close to 1 when  $n$  is large – is not meant to enlighten us for small values of  $n$ . The sacrifice of accuracy discussed at the end of the solution to (b) is apparent for small  $n$ , but insignificant for large  $n$ , as long as all we need to know is that for large  $n$ , the probability  $P(n)$  is very close to 1. Of course, if we wanted to know something else about  $P(n)$ , this estimate might not be as useful.

As you might expect, if we had been more careful with our estimates, we could have obtained a result that says more about  $P(n)$  (and perhaps works better for some smaller values of  $n$ ). This would likely come with a harder proof. For example, do you think it is possible to find a constant  $d$  so that  $P(n) > 1 - \frac{d}{n^2}$  for all *sufficiently large*  $n$ ? If so, it would give an even better understanding of how  $P(n)$  behaves for large  $n$ , but would require much more careful estimates than those in part (b).