



Problem of the Month

Problem 2: November 2021

Problem

A *lattice point* is a point (a, b) in the plane with the property that a and b are both integers. In this problem, we will say that a lattice point $P(a, b)$ is *visible* if $a > 0$, $b > 0$, and the line segment connecting P and the origin does not contain any lattice points other than P and the origin.

- How many lattice points $P(a, b)$ with $a \leq 10$ and $b \leq 10$ are visible?
- Determine the number of integers b with $b \leq 50$ for which $P(a, b)$ is visible when
 - $a = 6$
 - $a = 18$
 - $a = 36$.
- Determine how many points $P(a, b)$ with $a \leq 50$ and $b \leq 50$ are visible. There is quite a bit to do by hand, so you may want to use technology to help.
- Explain why the following equality is true:

$$\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{5^2}\right) \left(1 - \frac{1}{7^2}\right) \left(1 - \frac{1}{11^2}\right) \cdots = \frac{1}{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots}$$

The expressions on the left is the infinite product of all expressions of the form $1 - \frac{1}{p^2}$ where p is prime.

- It is well known that the infinite sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

is equal to $\frac{\pi^2}{6}$. This fact has many proofs and is originally due to the mathematician Leonhard Euler. You may wish to explore some of these proofs, but the intention in this problem is for you to take the result for granted.

Interestingly, the probability that a randomly chosen point in the first quadrant not on the axes is visible is $\frac{6}{\pi^2}$. Explain why this is true.

Note: It is ok to be a bit suspicious of what we mean by “probability” when choosing from an infinite set. Here is a way to think about what is meant in this problem: for a fixed positive integer, n , it is possible to compute the probability that a point $P(a, b)$ with $0 < a \leq n$ and $0 < b \leq n$ chosen randomly is visible. One might call this probability p_n . The question in (e), posed a bit more formally, might be “show that p_n gets very close to $\frac{6}{\pi^2}$ as n gets large”. If you have seen *limits*, you might want to formalize this further.



Hint

- (a) In all parts of this problem, it will be useful to think about how $P(a, b)$ being visible relates to $\gcd(a, b)$.
 - (b) All three parts have the same answer.
 - (c) For positive integers u and v , the number of positive multiples of u that are no larger than v is $\left\lfloor \frac{v}{u} \right\rfloor$. You may need to look up the notation $\lfloor x \rfloor$. If you are solving this problem by hand, you might want to first consider how many visible points there are of the form $P(a, b)$ when a is prime.
 - (d) What is the reciprocal of the sum of the geometric series $1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \dots$?
 - (e) What is the probability that two randomly chosen positive integers are both even? What is the probability that two randomly chosen integers are both multiples of 3?
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