Problem of the Month

Solution to Problem 2: November 2021

(a) Suppose $P(a,b)$ is visible. Since $a > 0$, we have that $a \neq 0$ and so the line segment connecting P to the origin has equation $y = \frac{b}{a}x$. Now suppose m is a positive common divisor of a and b. Then there are integers a' and b' with $0 < a' \leq a$, $0 < b' \leq b$, $a = a'm$, and $b = b'm$. Then $\frac{b}{a}a' = \frac{b'm}{a'm}a' = b'$, so (a', b') is on the line segment connecting P to the origin. Since P is visible, we cannot have $a' < a$, so $a' = a$ which means $m = 1$. We assumed that m was a positive common divisor of a and b and deduced that $m = 1$. Therefore, if $P(a,b)$ is visible, then $\gcd(a,b) = 1$.

Now suppose $P(a,b)$ is not visible. This means there is some lattice point (a', b') on $y = \frac{b}{a}x$ with $0 < a' < a$. This means $b' = \frac{b}{a}a'$ which can be rearranged to $ab' = a'b$. Since a, b, a', and b' are all integers, we have that the integer $a'b$ is a multiple of the integer a. If $\gcd(a,b) = 1$, then a' is a multiple of a. However, this cannot happen since $a' < a$. Therefore, if $P(a,b)$ is not visible, then $\gcd(a,b) \neq 1$.

We have shown that a point P that is not on the axes is visible exactly when $\gcd(a,b) = 1$. Therefore, counting the visible points $P(a,b)$ with $0 < a \leq 10$ and $0 < b \leq 10$ is the same as counting ordered pairs (a,b) with $0 < a \leq 10$ and $0 < b \leq 10$ such that $\gcd(a,b) = 1$.

The table below has rows indexed by the possible integer values of a from 1 through 10 inclusive and columns indexed by the values of b from 1 through 10 inclusive. The cell in the row corresponding to a and the column corresponding to b contains $\gcd(a,b)$.

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1	2
3	1	1	3	1	1	3	1	1	3	1
4	1	2	1	4	1	2	1	4	1	2
5	1	1	1	1	5	1	1	1	1	5
6	1	2	3	2	1	6	1	2	3	2
7	1	1	1	1	1	1	7	1	1	1
8	1	2	1	4	1	2	1	8	1	2
9	1	1	3	1	1	3	1	1	9	1
10	1	2	1	2	5	2	1	2	1	10

By the reasoning above, the number of visible points is equal to the number of 1's in the table above. There are 63 1's in the table, so there are 63 visible points $P(a,b)$ with $0 < a \leq 10$ and $0 < b \leq 10$.

(b) We can answer all three parts of this question at once. Factoring into primes, we have $6 = 2 \times 3$, $18 = 2 \times 3 \times 3$, and $36 = 2 \times 2 \times 3 \times 3$. The only prime numbers that divide 6 are 2 and 3, and the same is true of 18 and 36.

Therefore, for $a = 6$, $a = 18$, and $a = 36$, $\gcd(a, b) = 1$ exactly when 2 is not a divisor of b and 3 is not a divisor of b. This means that the answer to (i), (ii), and (iii) is equal to the number of integers b with $0 < b \leq 50$ that are neither a multiple of 2 nor a multiple of 3.

There are fifty integers b satisfying $0 < b \leq 50$ and exactly half of them are even. Thus, 25 of the values of b are multiples of 2. The largest multiple of 3 that is no larger than 50 is 48. This means the multiples of 3 between 1 and 50 inclusive are 3, 6, 9, and so on up to 3×16. Therefore, 16 of the values of b are multiples of 3.

Each of the totals computed in the previous paragraph, 25 and 16, includes the integers that are multiples of both 2 and 3. An integer is a multiple of both 2 and 3 exactly when it is a multiple of 6. This means the total $25 + 16 = 41$ is equal to the number of values of b that are either a multiple of 2 or a multiple of 3, but it overcounts by the number of multiples of 6 since both 25 and 16 account for the number of multiples of 6.

The largest multiple of 6 that is no larger than 50 is 48, which is equal to 6×8, so there are 8 multiples of 6 that are no larger than 50. Therefore, $25 + 16 - 8 = 33$ integers b with $0 < b \leq 50$ have the property that they are either a multiple of 2, a multiple of 3, or both. We are interested in the number of integers b with $0 < b \leq 50$ that are neither a multiple of 2 nor a multiple of 3, which we can now compute as $50 - 33 = 17$.

For $a = 6$, $a = 18$, and $a = 36$, there are 17 visible points $P(a, b)$ with $0 < b \leq 50$.

(c) In this solution, we will compute the number of visible points with $0 < a \leq 50$ and $0 < b \leq 50$, though some of the calculations will not be shown. With that said, the author promises that the calculation was done entirely by hand, but will not deny that a calculator was used to check them.

By the reasoning from the beginning of the solution to part (a), we wish to count all pairs (a, b) with $\gcd(a, b) = 1$, $0 < a \leq 50$, and $0 < b \leq 50$. We will declare now that a and b are integers satisfying $0 < a \leq 50$ and $0 < b \leq 50$ in this solution to avoid repeating this quantification.

For a fixed a, the number of b with $\gcd(a, b) = 1$ is equal to the number of b such that b has no prime factors in common with a. Therefore, as we saw in part (b), the primes occurring in the prime factorization of a is what matters, not the number of times each prime occurs. To compute this in general, we will compute the number of integers b that *do* have a prime factor in common with a, then subtract the result from 50.

In general, we will need a way to compute the number of multiples of an integer n that are less than or equal to 50. Suppose k is the largest positive integer such that $kn \leq 50$. Then k is the number of multiples of n that are no larger than 50. In other words, what we seek is a general way to compute k from n. To do this, we observe that if k is the largest positive integer such that $kn \leq 50$, then $50 < (k + 1)n$, so we have $kn \leq 50 < (k + 1)n$. Dividing through by n gives $k \leq \dfrac{50}{n} < k + 1$. The quantities k and $k + 1$ are consecutive integers, so we conclude that k is the largest integer that is no larger than $\dfrac{50}{n}$. For example, with $n = 6$, we get that $\dfrac{50}{6} = 8.3333\ldots$, so the largest integer that is no larger than $\dfrac{50}{6}$ is 8. This agrees with what was found in part (b) since there we showed that there are 8 positive multiples of 6 that are no larger than 50.

We now introduce some standard notation. For a real number x, we denote by $\lfloor x \rfloor$ the largest integer that is less than or equal to x. For example, $\lfloor \pi \rfloor = 3$. When x is an integer, $\lfloor x \rfloor = x$. For example, $\lfloor 5 \rfloor = 5$.

We now state a general fact: If u and v are positive integers, then the number of positive multiples of u that are less than or equal to v is $\left\lfloor \frac{v}{u} \right\rfloor$. You might want to think about why this is also true when $v < u$.

We will now continue to address the given question. To start, we will count the number of visible points $P(a, b)$ when a is a prime number.

If a is a prime number, then the number of integers b for which $\gcd(a, b) \neq 1$ is equal to the number of multiples of a between 1 and 50 inclusive. Therefore, if a is prime, then the number of visible points is $50 - \left\lfloor \frac{50}{a} \right\rfloor$.

The prime numbers that are no larger than 50 are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$$

For the primes from $a = 29$ to $a = 47$, we have $25 < a < 50$, so $1 = \frac{50}{50} < \frac{50}{a} < \frac{50}{25} = 2$. Thus, for each of the six primes between 29 and 47 inclusive, it must be that $\left\lfloor \frac{50}{a} \right\rfloor = 1$, so there are $50 - 1 = 49$ visible points. This gives a total of $6 \times 49 = 294$ visible points.

For the other nine primes from $a = 2$ through $a = 23$, the table below has the value of a in the left column and the corresponding value of $50 - \left\lfloor \frac{50}{a} \right\rfloor$ in the right column.

a	$50 - \left\lfloor \frac{50}{a} \right\rfloor$
2	25
3	34
5	40
7	43
11	46
13	47
17	48
19	48
23	48

Since $50 - \left\lfloor \frac{50}{a} \right\rfloor$ is the number of visible points when a is prime, we can get the number of visible points with a prime by totaling the values in the right column of the table above and adding this total to 294. This gives $379 + 294 = 673$ visible points $P(a, b)$ when a is prime.

Next, consider the case when $a = 4$. Since $4 = 2^2$, an integer b has $\gcd(4, b) = 1$ exactly when $\gcd(2, b) = 1$. Put differently, b and 4 have a common divisor larger than 1 exactly when b and 2 have a common divisor larger than 1. The same is true of any positive integer power of 2. This means that if a is a power or 2, then there are the same number of visible points $P(a,b)$ as there are visible points $P(2,b)$. Thus, we get 25 visible points for each of $a=4$, $a=8$, $a=16$, and $a=32$. Likewise, the number of visible points when $a=9$ or $a=27$ is the same as when $a=3$, so there are 34 visible points for $a=9$ and $a=27$. By similar reasoning there are 40 visible points when $a=25=5^2$ and 43 visible points when $a=49=7^2$.

When $a=1$, $P(a,b)$ is always visible, so there are 50 visible points when $a=1$. We will now recap with a subtotal: if a is prime, a is a power of a prime, or $a=1$, then there are

$$673+4(25)+2(34)+40+43+50=974$$

visible points.

We still need to count the visible points when a takes the values

$$6,10,12,14,15,18,20,21,22,24,26,28,30,33,34,35,36,38,39,40,42,44,45,46,48,50$$

Using the reasoning in part (b), there are exactly 17 visible points $P(a,b)$ if the prime divisors of a are exactly 2 and 3. This accounts for a taking on the values

$$6,12,18,24,36,48$$

so we get $6\times 17=102$ visible points with the property that the prime divisors of a are exactly 2 and 3. We will do a more general computation before adding these to the running total of 974.

Generalizing the ideas to count the number of visible points when the prime divisors of a are exactly 2 and 3, suppose a is an integer with exactly two prime divisors, p and q. The number of integers less than or equal to 50 that are multiples of p is $\left\lfloor \frac{50}{p} \right\rfloor$ and the number of multiples of q is $\left\lfloor \frac{50}{q} \right\rfloor$. Each of these totals counts the common multiples of p and q, but since p and q are distinct primes, their common multiples are exactly the multiples of pq. Thus, the number of integers less than or equal to 50 that are multiples of either p or q is

$$\left\lfloor \frac{50}{p} \right\rfloor + \left\lfloor \frac{50}{q} \right\rfloor - \left\lfloor \frac{50}{pq} \right\rfloor$$

and hence, the number of visible points when the prime divisors of a are exactly p and q is

$$50-\left\lfloor \frac{50}{p} \right\rfloor - \left\lfloor \frac{50}{q} \right\rfloor + \left\lfloor \frac{50}{pq} \right\rfloor .$$

Indeed, with $p=2$ and $q=3$, we get

$$50-\left\lfloor \frac{50}{2} \right\rfloor - \left\lfloor \frac{50}{3} \right\rfloor + \left\lfloor \frac{50}{6} \right\rfloor = 50-25-16+8=17.$$

When $p=2$ and $q=5$, there are $50-\left\lfloor \frac{50}{2} \right\rfloor - \left\lfloor \frac{50}{5} \right\rfloor + \left\lfloor \frac{50}{10} \right\rfloor = 50-25-10+5=20$ visible points. This is the number of visible points for $a=2\times 5=10$, $a=2^2\times 5=20$, $a=2^3\times 5=40$, and $a=2\times 5^2=50$.

When $p = 2$ and $q = 7$, there are $50 - 25 - 7 + 3 = 21$ visible points, which gives the number of visible points when $a = 14$, $a = 28$, and $a = 42$.

In the table below, the numbers of visible points when a has exactly two prime divisors are summarized. There are four columns in the table. In each row, the cell in the first column contains a prime p, the cell in the second column contains a prime q with $p < q$, the cell in the third column contains the number of visible points for any a whose prime divisors are exactly p and q, and the cell in the fourth column contains a list of the values of a with exactly these two prime divisors. Therefore, to find the number of visible points $P(a, b)$ if a has exactly two prime divisors p and q, locate a in the fourth column and the number of visible points will be the integer in the same row in the third column. Note that pairs (p, q) of primes are accounted for in the table below only if pq has at least one multiple less than or equal to 50.

p	q	$50 - \left\lfloor \frac{50}{p} \right\rfloor - \left\lfloor \frac{50}{q} \right\rfloor + \left\lfloor \frac{50}{pq} \right\rfloor$	a values
2	3	17	$6, 12, 18, 24, 36, 48$
2	5	20	$10, 20, 40, 50$
2	7	21	$14, 28$
2	11	23	$22, 44$
2	13	23	26
2	17	24	34
2	19	24	38
2	23	24	46
3	5	27	$15, 45$
3	7	29	21
3	11	31	33
3	13	32	39
5	7	34	35

The table above contains the number of visible points for every remaining a other than $a = 30$ and $a = 42$. Adding the totals for each of these 24 values of a, we get

$$(6 \times 17) + (4 \times 20) + (2 \times 21) + (3 \times 23) + (3 \times 24) + (2 \times 27) + 29 + 31 + 32 + 34 = 545$$

Adding to our previous total, we get that there are $974 + 545 = 1519$ visible points $P(a, b)$ where a is either 1, is a prime, is a power of a prime, or has exactly two distinct prime divisors. As mentioned above, we now have counted the visible points except for when $a = 30$ and $a = 42$. Notice that these are the only two positive integers less than 50 that have more than two distinct prime divisors.

You may wish to think about a general way to count the number of visible points where a has exactly three distinct prime divisors. It may be useful to read about the *inclusion-exclusion principle*. For this solution, we will just list the values of b that have $\gcd(30, a) = 1$ and $\gcd(42, b) = 1$, respectively. For 30, they are

$$1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49$$

and for 42 they are

$$1, 5, 11, 13, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47$$

for totals of 14 visible points for each of $a = 30$ and $a = 42$. Therefore, the total number of visible points $P(a, b)$ with $0 < a \le 50$ and $0 < b \le 50$ is $1519 + 14 + 14 = 1547$.

(d) To get an idea of why this is true, we first consider the following even more suspicious looking expression:

$$\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\cdots\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\cdots\right)\left(1+\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\cdots\right)\cdots$$

This is a product of infinitely many sums. Each sum is an infinite sum of the reciprocals of the *even* powers of a prime.

Although it may require some imagination, consider what would happen if we were to multiply this expression out. Each "term" would be a product of one summand from each parenthetical expression. Suppose, for example, we choose the $\frac{1}{p^2}$ term for each prime p. Then we would get a term

$$\frac{1}{2^2 \times 3^2 \times 5^2 \times 7^2 \times 11^2 \times 13^2 \times 17^2 \times \cdots}$$

and since there are infinitely many primes, the denominator is a product of infinitely many numbers that are greater than 1. This cannot possibly be any finite number, so we can interpret this term as $\frac{1}{\infty}$, which we have little choice but to interpret as being equal to 0.

We run into the same issue any time infinitely many of the "choices" are not equal to 1. Thus, for a term to "contribute" anything to the sum, we can only choose finitely many terms that are different from 1. That is, the expression above is equal to the sum of all terms obtained by choosing a term from each parenthetical expression so that only finitely many of the choices are different from 1. For example,

$$\frac{1}{2^2} \times \frac{1}{7^2} \times \frac{1}{19^6} = \frac{1}{(2 \times 7 \times 19^3)^2}$$

and

$$\frac{1}{7^8} \times \frac{1}{1009^2} = \frac{1}{(7^4 \times 1009)^2}$$

are terms in the sum.

If you think about it every term in the sum will be of the form $\frac{1}{n^2}$. Moreover, given a positive integer n, the prime factorization of n^2 has the form $p_1^{2e_1} p_2^{2e_2} \cdots p_k^{2e_k}$ where $p_1, \ldots, p_k$ are distinct primes and $e_1, \ldots, e_k$ are positive integers. By choosing $\frac{1}{p_i^{2e_i}}$ from the parenthetical expression for the prime p_i and 1 for all others, we get $\frac{1}{n^2}$ as a term in the sum. Since prime factorizations are unique, there is only one way that $\frac{1}{n^2}$ can arise as a term in the sum.

Therefore, it makes some sense that the product of infinite sums above is equal to the sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots.$$

Each sum in parentheses above is a geometric series, so we have the following:

$$\begin{aligned}
&1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots \\
&=\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\cdots\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\cdots\right)\left(1+\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\cdots\right)\cdots \\
&=\left(\frac{1}{1-\frac{1}{2^2}}\right)\left(\frac{1}{1-\frac{1}{3^2}}\right)\left(\frac{1}{1-\frac{1}{5^2}}\right)\left(\frac{1}{1-\frac{1}{7^2}}\right)\left(\frac{1}{1-\frac{1}{11^2}}\right)\cdots
\end{aligned}$$

which shows that

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\left(\frac{1}{1-\frac{1}{2^2}}\right)\left(\frac{1}{1-\frac{1}{3^2}}\right)\left(\frac{1}{1-\frac{1}{5^2}}\right)\left(\frac{1}{1-\frac{1}{7^2}}\right)\left(\frac{1}{1-\frac{1}{11^2}}\right)\cdots$$

Now take reciprocals of both sides of the equation above to get

$$\frac{1}{1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\cdots$$

(e) It was argued at the beginning of the solution to part (a) that a point $P(a,b)$ is visible exactly when $\gcd(a,b)=1$. Equivalently, $P(a,b)$ is visible exactly when a and b have no prime divisors in common.

We will first discuss the prime $p=2$. Suppose a is chosen randomly in the range $0<a\leq n$ for some integer n. If n is even, then there is exactly a $\frac{1}{2}$ chance that a is a multiple of 2. If n is odd, then there is a $\frac{\frac{n-1}{2}}{n}=\frac{1}{2}-\frac{1}{2n}$ chance that a is a multiple of 2. Notice that in the latter case, the probability is very close to $\frac{1}{2}$ when n is large since the quantity $\frac{1}{2n}$ is close to 0. Thus, if n is large, the probability that a is a multiple of 2 is extremely close to $\frac{1}{2}$, whether n is even or odd.

Now suppose a and b are both between 1 and n inclusive. The probability that a and b are both multiples of 2 is close to $\frac{1}{2}\times\frac{1}{2}$, and this implies that the probability that a and b do *not* have a common divisor of 2 is close to $1-\frac{1}{2^2}$.

We point out that $1-\frac{1}{2^2}$ becomes a better estimate for the probability as n gets larger.

More generally, consider a prime p and some large fixed positive integer, n. Now choose an integer a randomly with $0<a\leq n$. If n happens to be a multiple of p, then the probability that a is a multiple of p is exactly $\frac{1}{p}$. If n is not a multiple of p, then the probability that a is a multiple of p is close to $\frac{1}{p}$ (as with $p=2$, it gets closer as n gets larger). By the same reasoning as with $p=2$, if n is large and $P(a,b)$ is chosen randomly with $0<a\leq n$ and $0<b\leq n$, then the probability that a and b do not have a common divisor of p is close to $1-\frac{1}{p^2}$.

Now consider two different prime numbers p and q. Following reasoning similar to that which is above, if a point $P(a, b)$ is chosen randomly, there is a $\frac{1}{p^2}$ chance that a and b have a common divisor of p, there is a probability of $\frac{1}{q^2}$ that a and b have a common divisor of q, and there is a probability of $\frac{1}{(pq)^2}$ that a and b have a common divisor of both p and q. The latter probability is because a number is a multiple of p and q exactly when it is a multiple of pq.

We can now say that the probability that a and b have either a divisor of p in common and a divisor of q in common should be very close to

$$\frac{1}{p^2} + \frac{1}{q^2} - \frac{1}{(pq)^2}$$

where we subtract $\frac{1}{(pq)^2}$ since it is the probability that an integer is a multiple of both p and q. Therefore, the probability that a and b have neither a divisor of p in common nor a divisor of q in common is

$$1 - \frac{1}{p^2} - \frac{1}{q^2} + \frac{1}{(pq)}^{2} = \left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{q^2}\right).$$

Now let Event 1 be the event that a and b do not have a common divisor of p and Event 2 be the event that a and b do not have a common divisor of q. We have shown that the probability that both Event 1 and Event 2 occur is equal to the product of the probabilities that the events occur individually. In probability theory, we would conclude that Event 1 and Event 2 are independent.

The probability that $\gcd(a, b) = 1$ is equal to the probability that a and b have no prime divisors in common. By the reasoning above, the probability that they have no prime divisors in common is the product of the probabilities for each individual prime. Therefore, the probability is

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\cdots\left(1 - \frac{1}{p^2}\right)\cdots$$

In part (d), we argued that this quantity is the reciprocal of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$. It was given in part (e) that $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$. Thus, the probability is $\frac{6}{\pi^2} \approx 0.6079$.

Below is a simple Python script that, for a given positive integer n, returns the proportion of points $P(a, b)$ with $0 < a \leq n$ and $0 < b \leq n$ that are visible.

```
import math
n = int(input())
count = 0
for a in range(1,n+1):
    for b in range(1,n+1):
        if math.gcd(a,b) == 1:
            count += 1
print(float(count)/n**2)
```

Final Remark: You may wonder if a similar analysis can be performed in 3 dimensions. That is, we can consider points in space with coordinates (a, b, c) that are all positive integers and ask whether it is visible. Here, a point being "visible" would again mean there are no points with integer coordinates on the line segment connecting it to the origin.

In fact, a very similar argument can be used to relate the probability that a point (a, b, c) with positive integer coordinates is visible to the quantity

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \cdots .$$

An interesting fact about the quantity above is that there is no known "closed form" like there is for the sum of the reciprocals of the squares. It is known, however, that it does "equal" something, and that value is about 1.2020569.

You might even stretch your imagination further to wonder about the probability that a point with positive integer coordinates is "visible" in four or more dimensions. It turns out to always be related to the quantity

$$1 + \frac{1}{2^d} + \frac{1}{3^d} + \frac{1}{4^d} + \frac{1}{5^d} + \cdots$$

where d is the dimension. For $d \geq 2$, this quantity always "equals" something, and in fact, it is known that when d is even, it is equal to π^d times a rational number. You may wish to search "Bernoulli numbers" for more information.

These infinite sums are special values of a very famous function known as the "Riemann Zeta Function". This function has been an important object of study in mathematics for well over a century and there are still many unsolved problems involving it.