Problem of the Month
## Problem 3: December 2021

Problem

Before stating the problem, we will introduce some notation and terminology.

- In $\triangle ABC$, we will denote the length of BC by a, the length of AC by b, and the length of AB by c.

- The *semiperimeter* of $\triangle ABC$ will be denoted by s and is equal to $\dfrac{a+b+c}{2}$.

- The *incircle* of $\triangle ABC$ is the unique circle that is tangent to all three sides of $\triangle ABC$. Its radius is called the *inradius* of $\triangle ABC$ and is denoted by r. An important fact about the incircle is that its centre is at the intersection of the three angle bisectors of the triangle.

- The *circumcircle* of $\triangle ABC$ is the unique circle on which all three of A, B, and C lie. Its radius is called the *circumradius* of $\triangle ABC$ and is denoted by R. An important fact about the circumcircle is that its centre is at the intersection of the perpendicular bisectors of the three sides of the triangle.

The diagram below illustrates some of the information above.
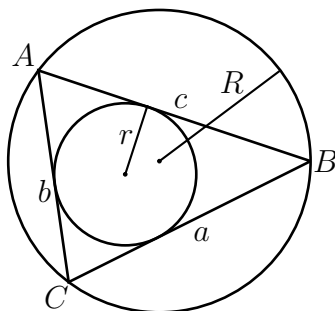
This problem is about right-angled triangles. Most of us are aware of the famous Pythagorean theorem, but there are other interesting properties only satisfied by right-angled triangles.

(a) Suppose $\triangle ABC$ is right-angled at C and that h is the length of the altitude from C to AB. Show that $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{h^2}$.

(b) Suppose $\triangle ABC$ is right-angled. Show that $\cos^2 \angle A + \cos^2 \angle B + \cos^2 \angle C = 1$.

(c) Suppose $\triangle ABC$ is right-angled. Show that $s = r + 2R$.

(d) Suppose $\triangle ABC$ satisfies $a^2 + b^2 = c^2$. Prove that $\angle C = 90°$.

(e) Suppose $\triangle ABC$ satisfies $\cos^2 \angle A + \cos^2 \angle B + \cos^2 \angle C = 1$. Prove that $\triangle ABC$ is right-angled.

(f) Suppose $\triangle ABC$ satisfies $s = r + 2R$. Show that $\triangle ABC$ is right-angled. [A solution to this problem will likely require some general identities involving the inradius and circumradius. Some specific useful identities will be given in the hint.]

Hint

There are several ways to approach each of the parts of this problem. The hints below correspond to the solutions that will be provided. You might find solutions that do not use the ideas in these hints.

(a) Apply the Pythagorean theorem to some of the smaller right-angled triangles that appear once the altitude is drawn.

(b) If there is a right angle at A, then what does the equation become?

(c) The important facts in the problem statement about how to find the centres of the incircle and circumcircle may be useful.

(d) Be careful not to confuse the statements "*If $\angle C = 90°$ then $a^2 + b^2 = c^2$*" and "*If $a^2 + b^2 = c^2$ then $\angle C = 90°$*". The first statement is what is usually considered the Pythagorean theorem. The second statement is its *converse*, and this is the statement this problem asks you to verify. This means that you cannot assume that $\angle C = 90°$ has a right angle; you need to assume $a^2 + b^2 = c^2$ and deduce that $\angle C = 90°$.

(e) Try to use trigonometric identities to prove that $\big(\cos A\big)\big(\cos B\big)\big(\cos(A + B)\big) = 0$.

(f) We found several solutions to this problem and each of them involves significant algebraic manipulation. The simplest solution that we found involved an expression for $8R^2$ in terms of a, b, and c. In our solution, we will use the following two facts that are true of every triangle.

- The quantities rs, $\dfrac{abc}{4R}$, and $\sqrt{s(s-a)(s-b)(s-c)}$ are all equal to the area of $\triangle ABC$.

- The Law of Sines can be extended to the following set of equations:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

.

Proofs of these facts will not be included in the solution, but they can be easily found online. Better yet, try to prove them for yourself!