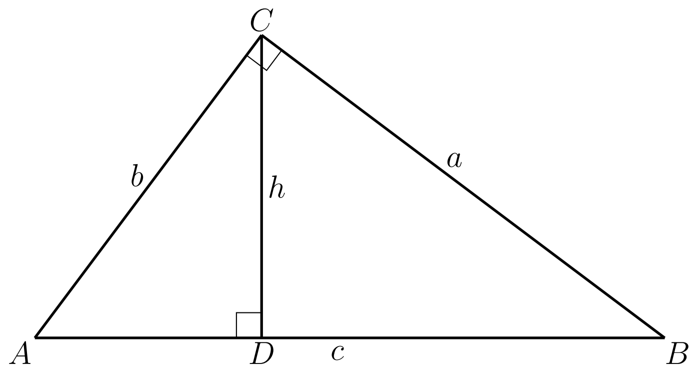
Problem of the Month

Solution to Problem 3: December 2021

(a) Suppose the altitude from C intersects side AB at D.



Because $\triangle ADC$ and $\triangle ACB$ are both right-angled and have $\angle A$ in common, we have that $\triangle ADC$ is similar to $\triangle ACB$. This means $\dfrac{DC}{AC} = \dfrac{CB}{AB}$ or $\dfrac{h}{b} = \dfrac{a}{c}$.

Rearranging this equation, we have

$$\frac{1}{h} = \frac{c}{ab}$$

Squaring both sides and using $c^2 = a^2 + b^2$, we have

$$\begin{aligned}\frac{1}{h^2} &= \frac{c^2}{a^2b^2} \\ &= \frac{a^2+b^2}{a^2b^2} \\ &= \frac{a^2}{a^2b^2} + \frac{b^2}{a^2b^2} \\ &= \frac{1}{b^2} + \frac{1}{a^2}\end{aligned}$$

(b) We will assume that $\angle C = 90^\circ$. The argument is similar if $\angle A = 90^\circ$ or $\angle B = 90^\circ$.

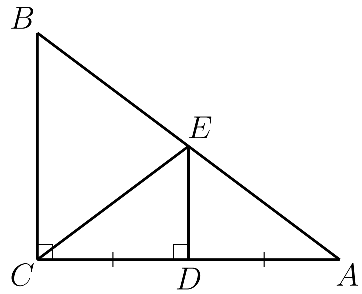
Since $\triangle ABC$ is right-angled at C, we have that $\cos\angle C = \cos 90^\circ = 0$. Therefore,

$$\begin{aligned}\cos^2\angle A + \cos^2\angle B + \cos^2\angle C &= \left(\frac{b}{c}\right)^2 + \left(\frac{a}{c}\right)^2 + 0^2 \\ &= \frac{a^2+b^2}{c^2} \\ &= \frac{c^2}{c^2} \\ &= 1\end{aligned}$$

(c) As with part (b), we will assume that the right angle occurs at C since the argument is similar if it is at A or B.

We will first show that the length of the hypotenuse is $2R$ by showing that the centre of the circumcircle is the midpoint of the hypotenuse.

Assume that D is the midpoint of AC. Let E be the point where the perpendicular bisector of AC intersects the hypotenuse of $\triangle ABC$. As well, connect E to C.

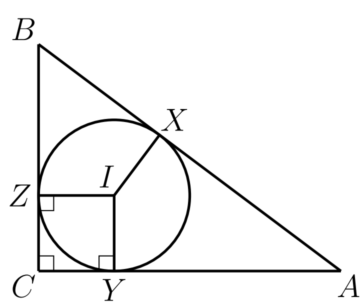


Since they share $\angle A$ and both have a right angle, $\triangle ABC$ and $\triangle AED$ are similar. Since $CD = AD$, we have that $\frac{AD}{AC} = \frac{1}{2}$, so $\frac{AE}{AB} = \frac{1}{2}$ because $\triangle ABC$ is similar to $\triangle AED$. Rearranging, we have $AB = 2AE$, which implies that E is the midpoint of AB.

Since $CD = AD$, $\angle CDE = \angle ADE = 90^\circ$, and $\triangle ADE$ and $\triangle CDE$ share side DE, we have that $\triangle ADE$ is congruent to $\triangle CDE$ by side-angle-side congruence. This means $CE = AE$. We now have that E is the midpoint of the hypotenuse of $\triangle ABC$ which means that $AE = BE$. Since $CE = AE$ as well, we have shown that the midpoint of the hypotenuse is equidistant from the three vertices of $\triangle ABC$.

If we draw a circle centred at E with radius equal to AE, it will pass through all three vertices of the triangle. The circumcircle always exists and is the only circle with this property, so this circle is in fact the circumcircle, which means $AE = BE = CE = R$. It follows that the length of the hypotenuse in a right-angled triangle is equal to $2R$.

The next image depicts a right-angled triangle with its incircle, having centre I.



A tangent to a circle is perpendicular to the radius connecting the centre to the point of tangency. Therefore, $\angle IYC = \angle IZC = 90^\circ$. Since $IYCZ$ is a quadrilateral with three right angles, its fourth angle must also be right, so $IYCZ$ is a rectangle. As well, $IZ = IY = r$, which means $IYCZ$ is a square with side length r. Thus, $CZ = CY = r$.

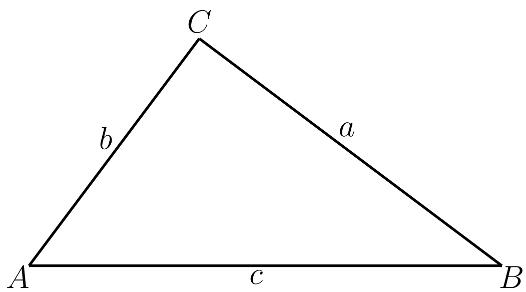
In general, if two tangents to the same circle intersect at some point outside the circle, then the distances from that point to each of the points of tangency are equal. In our case, this implies $BZ = BX$ and $AX = AY$. Using that $BZ = BX$, $AX = AY$, and

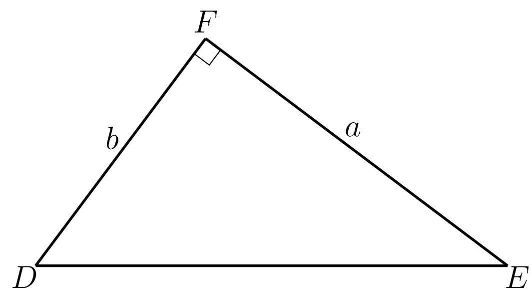
$CZ = CY = r$, we get that the perimeter of $\triangle ABC$ is

$$\begin{aligned} AX + AY + BX + BZ + CY + CZ &= AX + AX + BX + BX + r + r \\ &= 2(AX + BX + r) \\ &= 2(AB + r). \end{aligned}$$

Finally, since we also know that $AB = 2R$, we get that the perimeter of $\triangle ABC$ is $2(2R+r)$. Therefore, $s = r + 2R$.

(d) Assume that $a^2 + b^2 = c^2$ and then construct $\triangle DEF$ so that $DF = b$, $EF = a$, and $\angle DFE = 90°$.





By the Pythagorean theorem applied to $\triangle DEF$ (which is right-angled by construction), we have that $a^2 + b^2 = DE^2$. Since $a^2 + b^2 = c^2$, we have $c^2 = DE^2$. Since both c and DE are positive, it follows that $DE = c$. This means $\triangle ABC$ and $\triangle DEF$ are congruent by side-side-side congruence. Therefore, $\angle ACB = \angle DFE = 90°$, so $\triangle ABC$ is right-angled at C.

(e) In this solution, we will use the following identities that hold for all angles θ, x, and y.

$$2\cos^2\theta - 1 = \cos(2\theta) \qquad (1)$$

$$\cos(360° - \theta) = \cos\theta \qquad (2)$$

$$2\cos(x+y)\cos(x-y) = \cos(2x) + \cos(2y) \qquad (3)$$

$$\cos(x+y) + \cos(x-y) = 2\cos x\cos y \qquad (4)$$

We will begin with the assumption that $\cos^2\angle A + \cos^2\angle B + \cos^2\angle C = 1$ and deduce several equivalent identities. In an effort to declutter the calculation below, we will drop the "$\angle$" from $\angle A$, $\angle B$, and $\angle C$ and denote them by A, B, and C, respectively. In the calculation that follows, we will use the identities above by referring to their label of (1), (2), (3), or (4). The line labelled by (5) is using the fact that $A + B + C = 180°$.

$$\begin{aligned}
\cos^2 A + \cos^2 B + \cos^2 C &= 1 \\
\frac{\cos 2A + 1}{2} + \frac{\cos 2B + 1}{2} + \frac{\cos 2C + 1}{2} &= 1 && (1) \\
(\cos 2A + 1) + (\cos 2B + 1) + (\cos 2C + 1) &= 2 \\
\cos 2A + \cos 2B + \cos 2C &= -1 \\
\cos 2A + \cos 2B + \cos(2(180° - A - B)) &= -1 && (5) \\
\cos 2A + \cos 2B + \cos(2A + 2B) &= -1 && (2) \\
2\cos(A + B)\cos(A - B) + \cos\big(2(A + B)\big) &= -1 && (3) \\
2\cos(A + B)\cos(A - B) + 2\cos^2(A + B) - 1 &= -1 && (1) \\
2\cos(A + B)\cos(A - B) + 2\cos^2(A + B) &= 0 \\
2\cos(A + B)\big(\cos(A - B) + \cos(A + B)\big) &= 0 \\
4\cos(A + B)\cos A \cos B &= 0 && (4)
\end{aligned}$$

This means that either $\cos\angle A = 0$, $\cos\angle B = 0$, or $\cos(\angle A + \angle B) = 0$. If $\cos\angle A = 0$, then $\angle A = 90°$. If $\cos\angle B = 0$, then $\angle B = 90°$. If $\cos(\angle A + \angle B) = 0$, then $\angle A + \angle B = 90°$, which implies $\angle C = 180° - \angle A - \angle B = 90°$. In all three cases, $\triangle ABC$ has a right angle. Note that $\angle A$, $\angle B$, $\angle C$, and $\angle A + \angle B$ all measure between $0°$ and $180°$, so a cosine of 0 does imply an angle of $90°$.

(f) As mentioned in the hint, the area of $\triangle ABC$ is equal to rs as well as $\frac{abc}{4R}$. Equating these two expressions gives $rs = \frac{abc}{4R}$ which can be rearranged to get

$$4rR = \frac{abc}{s} \tag{$*$}$$

By Heron's formula, the area of the triangle is also equal to $\sqrt{s(s-a)(s-b)(s-c)}$. This implies $rs = \sqrt{s(s-a)(s-b)(s-c)}$, so we can square both sides and solve to get

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} \tag{$**$}$$

Starting with the equation, $s = r + 2R$, we can square both sides to get $s^2 = r^2 + 4rR + 4R^2$. We will now solve for $8R^2$ in terms of a, b, and c using this equation, some algebraic manipulation, as well as $(*)$ and $(**)$ above.

$$\begin{aligned}
4R^2 &= s^2 - r^2 - 4rR \\
4R^2 &= s^2 - \frac{(s-a)(s-b)(s-c)}{s} - \frac{abc}{s} && (*), (**) \\
8R^2 &= 2s^2 - \frac{2(s-a)(s-b)(s-c)}{s} - \frac{2abc}{s} \\
&= \frac{1}{s}\big(2s^3 - 2(s-a)(s-b)(s-c) - 2abc\big) \\
&= \frac{1}{s}\big(2s^3 - 2(s^3 - s^2(a+b+c) + s(ab+ac+bc) - abc) - 2abc\big) \\
&= \frac{1}{s}\big(2s^2(a+b+c) - 2s(ab+ac+bc)\big) \\
&= 2s(a+b+c) - 2(ab+ac+bc)
\end{aligned}$$

Now note that $2s = a+b+c$, so in fact

$$\begin{aligned}
8R^2 &= (a+b+c)^2 - 2(ab+ac+bc) \\
&= a^2+b^2+c^2+2(ab+ac+bc) - 2(ab+ac+bc) \\
&= a^2+b^2+c^2.
\end{aligned}$$

In the hint, the *Extended Law of Sines* was given and says that

$$\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C} = 2R$$

for any triangle. This implies the following three equations

$$\frac{a^2}{4R^2} = \sin^2\angle A \qquad \frac{b^2}{4R^2} = \sin^2\angle B \qquad \frac{c^2}{4R^2} = \cos^2\angle C.$$

Dividing $8R^2 = a^2+b^2+c^2$ by $4R^2$, we get

$$\begin{aligned}
2 &= \frac{a^2}{4R^2} + \frac{b^2}{4R^2} + \frac{c^2}{4R^2} \\
&= \sin^2\angle A + \sin^2\angle B + \sin^2\angle C \\
&= (1-\cos^2\angle A) + (1-\cos^2\angle B) + (1-\cos^2\angle C)
\end{aligned}$$

which implies $2 = 3 - (\cos^2\angle A + \cos^2\angle B + \cos^2\angle C)$. This equation can be rearranged to get $\cos^2\angle A + \cos^2\angle B + \cos^2\angle C = 1$.

We have now assumed that $s = r + 2R$ and deduced that $\cos^2\angle A + \cos^2\angle B + \cos^2\angle C = 1$. By part (e), $\triangle ABC$ must be right-angled.