Problem of the Month

Problem 4: January 2022

Problem

The goal of this problem is to work through some techniques that can sometimes help find the roots of polynomials. The statements of some parts of this problem refer to *repeated roots*, which we will now define. Suppose r is a root of the polynomial $p(x)$, that is, $p(r) = 0$. You may already know that if $p(r) = 0$, then $(x - r)$ divides evenly into $p(x)$. We say that r is a repeated root of $p(x)$ if $(x - r)^2$ divides evenly into $p(x)$. For example, 1 is a repeated root of $x^2 - 2x + 1$ because $x^2 - 2x + 1 = (x - 1)^2$, and 2 is a repeated root of $x^4 - 5x^3 + 6x^2 + 4x - 8$ since $x^4 - 5x^3 + 6x^2 + 4x - 8 = (x - 2)^2(x^2 - x - 2)$.

(a) The polynomials $p(x) = 2x^2 - 1275x + 194292$ and $q(x) = x^2 - 635x + 96516$ have a root in common. Determine both roots of both polynomials without using the quadratic formula.

(b) Let $p(x) = x^3 + ax^2 + bx + c$ be a polynomial with a root r. Show that r is a repeated root of $p(x)$ if and only if r is a root of the polynomial $q(x) = 3x^2 + 2ax + b$.

You may recognize $q(x)$ as the *derivative* of $p(x)$. If you are familiar with derivatives, you might want to try to generalize this part.

(c) Suppose $p(x) = x^3 + bx + c$ has roots u, v, and w (which may not all be different). Express the quantity $(u - v)^2(v - w)^2(w - u)^2$ in terms of b and c. This quantity is known as the discriminant of $p(x)$, and this exercise shows that its value can be determined from the coefficients without knowing the roots. Explain how, without knowing any of the roots, it is possible to determine if a cubic of the form $x^3 + bx + c$ has a repeated root.

(d) Consider the polynomial $p(x) = x^3 + ax^2 + bx + c$. Show that the coefficient of x^2 in the polynomial $q(x) = p\left(x - \frac{a}{3}\right)$ is equal to 0. Explain how the roots of $p(x)$ can be found easily if the roots of $q(x)$ are known.

(e) Find all roots of the polynomial $p(x) = x^3 - 135x^2 + 5832x - 81648$.

Hint

In several parts of this problem, it might be useful to review how the roots of a polynomial are related to its coefficients. In particular, how the coefficients arise as combinations of the roots.

(a) Suppose r is a common root of $p(x)$ and $q(x)$. Can you use what you know about $p(r)$ and $2q(r)$ to deduce the value of r without using the quadratic formula?

(b) Assuming there is a repeated root, use the relationships between the roots and the coefficients to compute $q(r)$ where r is the repeated root. For the other direction, you can try something similar or attempt to divide $p(x)$ by $(x-r)^2$, where r is the common root of $p(x)$ and $q(x)$.

(c) Try to show that $(u-v)^2 = -ab - 3uv$, and do not forget what it means for u, v, and w to be roots of $p(x)$!

(d) Think about how the graphs of the polynomials $p(x)$ and $q(x)$ compare to one another.

(e) First use part (d) to find a cubic that has its coefficient of x^2 equal to 0, but whose roots are a translation of those of $p(x)$. Next, use part (c) to show that this new polynomial, and hence, the original one, has a repeated root. Next, use part (b) to find a quadratic that shares that repeated root, and finally use the idea from part (a) to find the repeated root.