Problem of the Month
Problem 5: February 2022

In each part of this problem, there is a hallway containing of K doors numbered consecutively from 1 to K that are all initially closed. To *toggle* a door means to open it if it is closed and to close it if it is open. We will also use the notation that for a positive integer n, $\tau(n)$ is equal to the number of positive integer factors of n. For example, $\tau(1) = 1$ since 1 has exactly one positive factor and for a prime number p, we always have $\tau(p) = 2$ since prime numbers have exactly two positive factors. For another example $\tau(10) = 4$ since it has four positive integer factors, 1, 2, 5, and 10.

(a) In this part, $K = 100$. 100 "steps" are performed as follows:

- In step 1, every door that is numbered with a multiple of 1 is toggled.
- In step 2, every door that is numbered with a multiple of 2 is toggled.
- In step 3, every door that is numbered with a multiple of 3 is toggled.

In step n, every door that is numbered with a multiple of n is toggled. After all 100 steps are performed, which doors are open?

(b) In this part, $K = 100$. As with part (a), 100 steps are performed with one step for each integer n from 1 through K. This time, in step n, each door that is numbered with a multiple of n is toggled n times. For example, in step 5, each door that is numbered with a multiple of 5 is to be toggled 5 times. After all 100 steps are performed, which doors are open?

(c) In this part, $K = 2^9 \times 3^4 \times 5^{13} \times 7^{12}$. As with parts (a) and (b), a step is performed for each positive integer n from 1 through K. In step n, every door that is numbered by a multiple of n is toggled $\tau(n)$ times. For example, in step 5, every door that is numbered by a multiple of 5 is toggled $\tau(5) = 2$ times.

After all K steps are performed, is the door numbered with K open or closed?