Problem of the Month
Problem 5: February 2022

Problem

In each part of this problem, there is a hallway containing of K doors numbered consecutively from 1 to K that are all initially closed. To *toggle* a door means to open it if it is closed and to close it if it is open. We will also use the notation that for a positive integer n, $\tau(n)$ is equal to the number of positive integer factors of n. For example, $\tau(1) = 1$ since 1 has exactly one positive factor and for a prime number p, we always have $\tau(p) = 2$ since prime numbers have exactly two positive factors. For another example $\tau(10) = 4$ since it has four positive integer factors, 1, 2, 5, and 10.

(a) In this part, $K = 100$. 100 "steps" are performed as follows:

- In step 1, every door that is numbered with a multiple of 1 is toggled.
- In step 2, every door that is numbered with a multiple of 2 is toggled.
- In step 3, every door that is numbered with a multiple of 3 is toggled.

In step n, every door that is numbered with a multiple of n is toggled. After all 100 steps are performed, which doors are open?

(b) In this part, $K = 100$. As with part (a), 100 steps are performed with one step for each integer n from 1 through K. This time, in step n, each door that is numbered with a multiple of n is toggled n times. For example, in step 5, each door that is numbered with a multiple of 5 is to be toggled 5 times. After all 100 steps are performed, which doors are open?

(c) In this part, $K = 2^9 \times 3^4 \times 5^{13} \times 7^{12}$. As with parts (a) and (b), a step is performed for each positive integer n from 1 through K. In step n, every door that is numbered by a multiple of n is toggled $\tau(n)$ times. For example, in step 5, every door that is numbered by a multiple of 5 is toggled $\tau(5) = 2$ times.

After all K steps are performed, is the door numbered with K open or closed?

Hint

(a) Solving this problem will come down to counting how many positive factors each door number has. It might be useful to determine the number of factors of the first few positive integers and see if you notice a pattern. Try determining the number of factors of the integers from 1 through 20.

(b) In this part, whether a door is open or closed only depends on how many odd positive factors it has.

(c) The following general fact may be useful in this or the other parts: Every integer n can be expressed in the form $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where the p_i are distinct prime numbers and the e_i are positive integers. An integer d is a positive factor of n if and only if it can be expressed in the form $d = p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k}$ where $0 \leq f_i \leq e_i$ for each i. This means that $\tau(n) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$. Can you see why?