Problem of the Month

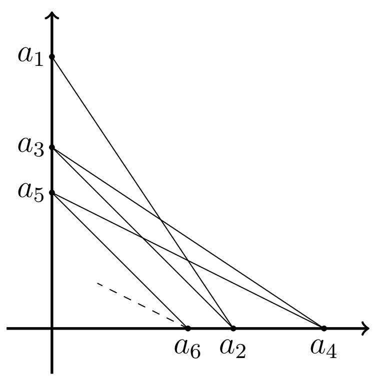
Problem 6: March 2022

Problem

In this problem, we will explore the following construction: Start with the positive real number $a_1 = 1$ and an infinite sequence $m_1, m_2, m_3, \ldots$ of negative slopes that are all distinct. For $n \geq 1$, we define a_{n+1} from a_n as follows.

- For odd n, a_{n+1} is the x-intercept of the line with slope m_n through $(0, a_n)$.
- For even n, a_{n+1} is the y-intercept of the line with slope m_n through $(a_n, 0)$.

The diagram below illustrates this. The line through $(0, a_1)$ and $(a_2, 0)$ has slope m_1, the line through $(a_2, 0)$ and $(0, a_3)$ has slope m_2, and so on.



(a) Suppose that $m_n = -\dfrac{1}{2^n}$ for all $n \geq 1$.

  (i) Compute a_2, a_3, a_4, and a_5.

  (ii) Find a general formula for a_n. You will likely need a separate formula for even n and odd n. Describe what happens to a_n as n gets large.

(b) Suppose that $m_n = -\dfrac{1}{2^{\frac{1}{2^n}+1}}$ for all n. [The exponent in the denominator is $\frac{1}{2^n} + 1$]

  (i) Find a general formula for a_n.

  (ii) Describe what happens to a_n as n gets large.

(c) Let u and v be arbitrary positive real numbers with $u \neq 1$. Give a sequence of slopes so that the sequence $a_1, a_3, a_5, a_7, \ldots$ approaches u and the sequence $a_2, a_4, a_6, a_8, \ldots$ approaches v. Remember that the sequence of slopes should not contain any repetitions.

(d) Suppose $m_n = -\dfrac{1}{n}$ for all $n \geq 1$.

  (i) Find an integer n so that $a_n < \dfrac{1}{100}$.

  (ii) Find an integer n so that $a_n > 100$.

Hint

Before attempting any of the problems, it might useful to show that a_{n+1} can be expressed in terms of a_n and m_n.

(b) In parts (i) and (ii), try computing the first few a_n and looking for a pattern. Do you notice a familiar type of series forming in the exponents?

(b) If you are comfortable with logarithms, you might find that it simplifies some calculations to define $A_n = \log_2(a_n)$ and work with the A_n instead. If you can find a general formula for A_n, then you can find a general formula for a_n by using that $a_n = 2^{A_n}$.

(c) Use the idea from part (b) to construct the sequence of slopes. What happens when you change the $\frac{1}{2^n} + 1$ in the exponent to $\frac{1}{2^n} + c$ for some $c \neq 1$?

(d) To start, find a general formula for a_n. A separate formula for even n and odd n will probably be useful. For odd n, try to show that $(a_n)^2$ is less than $\frac{1}{n-1}$. Can you do something similar for even n?