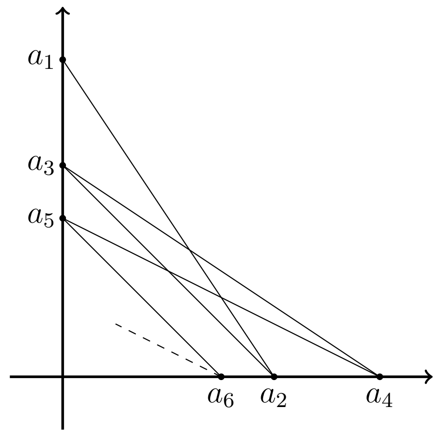
Problem of the Month
Solution to Problem 6: March 2022

As suggested in the hint, we will first show how a_{n+1} can be obtained directly from a_n and m_n. We will consider the even and odd cases separately. Consider the diagram from the problem statement:



When n is odd, we have by definition that the slope of the line through $(0, a_n)$ and $(a_{n+1}, 0)$ is m_n. This means $m_n = \dfrac{a_n - 0}{0 - a_{n+1}} = \dfrac{a_n}{-a_{n+1}}$. Solving for a_{n+1} gives $a_{n+1} = -\dfrac{a_n}{m_n}$.

When n is even, the slope of the line through $(0, a_{n+1})$ and $(a_n, 0)$ has slope m_n. This means $m_n = \dfrac{a_{n+1} - 0}{0 - a_n} = \dfrac{a_{n+1}}{-a_n}$, which implies $a_{n+1} = -a_n m_n$.

Putting these cases together, we get that

$$a_{n+1} = \begin{cases} -\dfrac{a_n}{m_n} & \text{if } n \text{ is odd} \\ -a_n m_n & \text{if } n \text{ is even} \end{cases} \tag{1}$$

Note that since the m_n are negative, they are non-zero. As well, $a_1 = 1$ is nonzero, so every a_n is nonzero since each is obtained from the previous by either multiplying or dividing by a non-zero slope.

(a) (i) Using Equation (1) and that $a_1 = 1$, we get

$$a_2 = -\frac{a_1}{m_1} = -\frac{1}{-\frac{1}{2}} = 2 \qquad a_3 = -a_2 m_2 = -2\left(-\frac{1}{2^2}\right) = \frac{1}{2}$$

$$a_4 = -\frac{a_3}{m_3} = -\frac{\frac{1}{2}}{-\frac{1}{2^3}} = 4 \qquad a_5 = -a_4 m_4 = -4\left(-\frac{1}{2^4}\right) = \frac{1}{4}$$

(ii) Based on the calculations in (i), we might guess that the sequence $a_1, a_2, a_3, a_4, \ldots$ is

$$1, 2, \frac{1}{2}, 2^2, \frac{1}{2^2}, 2^3, \frac{1}{2^3}, 2^4, \frac{1}{2^4}, \ldots$$

and after a bit of thought, this can be expressed more precisely as

$$a_n = \begin{cases} \dfrac{1}{2^{\frac{n-1}{2}}} & \text{if } n \text{ is odd} \\[2ex] 2^{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases} \tag{2}$$

We will verify that Equation (2) holds using *mathematical induction*. Using the calculations in part (i), it can be checked that Equation (2) holds when $n = 1$, $n = 2$, $n = 3$, $n = 4$, and $n = 5$. We will now show that if Equation (2) is true for some positive integer k, then it is true for $k+1$. By the principle of mathematical induction, this will imply that it is true for all positive integers.

Suppose k is a positive integer and that Equation (2) is true for $n = k$. We will consider two cases:

Case 1: k is odd.
Since Equation (2) holds for $n = k$, we have that $a_k = \dfrac{1}{2^{\frac{k-1}{2}}}$. By Equation (1), we also have that $a_{k+1} = -\dfrac{a_k}{m_k}$. Using these equations and that $m_k = -\dfrac{1}{2^k}$, we can calculate

$$a_{k+1} = -\frac{a_k}{m_k} = -\frac{\frac{1}{2^{\frac{k-1}{2}}}}{-\frac{1}{2^k}} = \frac{2^k}{2^{\frac{k-1}{2}}} = 2^{k-\frac{k-1}{2}} = 2^{\frac{k+1}{2}}$$

and since k is odd, $k+1$ is even, so the above calculation shows that Equation (2) holds when $n = k+1$.

Case 2: k is even.
Since Equation (2) holds for $n = k$, we have that $a_k = 2^{\frac{k}{2}}$. By Equation (1), we also have that $a_{k+1} = -a_k m_k$. Similar to Case 1, we get

$$a_{k+1} = -a_k m_k = -2^{\frac{k}{2}}\left(-\frac{1}{2^k}\right) = 2^{\frac{k}{2}-k} = \frac{1}{2^{\frac{k}{2}}} = \frac{1}{2^{\frac{(k+1)-1}{2}}}$$

and since $k+1$ is odd, this shows that Equation (2) holds for $k+1$.

As mentioned above, this establishes that Equation (2) holds for all positive integers n.

As n gets large, the terms a_n are getting larger and larger without bound for even n and are approaching 0 for odd n.

(b) (i) In this part, we will make use of logarithms to simplify some of the calculations. In fact, you might want to go back and try part (a) again using logarithms since they are useful there too!

Specifically, we will define $A_n = \log_2(a_n)$ for each $n \geq 1$. You may wish to spend some time convincing yourself that a_n is always positive which implies that there is no issue with taking its logarithm.

With A_n defined, we will apply Equation (1) with logarithm rules to get related equations for A_n. When n is odd, we get

$$\begin{aligned} A_{n+1} = \log_2(a_{n+1}) &= \log_2\left(-\frac{a_n}{m_n}\right) \\ &= \log_2(a_n) - \log_2(-m_n) \\ &= A_n - \log_2\left(\frac{1}{2^{\frac{1}{2^n}+1}}\right) \\ &= A_n + \log_2\left(2^{\frac{1}{2^n}+1}\right) \\ &= A_n + \frac{1}{2^n} + 1 \end{aligned}$$

When n is even, we get

$$\begin{aligned} A_{n+1} = \log_2(a_{n+1}) &= \log_2\left(a_n(-m_n)\right) \\ &= \log_2(a_n) + \log_2\left(\frac{1}{2^{\frac{1}{2^n}+1}}\right) \\ &= A_n - \frac{1}{2^n} - 1 \end{aligned}$$

Keep in mind that m_n is negative, so $-m_n$ is positive. Putting these equations together, we get

$$A_{n+1} = \begin{cases} A_n + \frac{1}{2^n} + 1 & \text{if } n \text{ is odd} \\ \\ A_n - \frac{1}{2^n} - 1 & \text{if } n \text{ is even} \end{cases} \tag{3}$$

Now observe that $A_1 = \log_2(a_1) = \log_2(1) = 0$. From this observation and Equation (3) we get

$$\begin{aligned} A_2 &= A_1 + \frac{1}{2} + 1 \\ &= \frac{1}{2} + 1 \end{aligned} \qquad \begin{aligned} A_3 &= A_2 - \frac{1}{2^2} - 1 \\ &= \frac{1}{2} - \frac{1}{2^2} \end{aligned}$$

$$\begin{aligned} A_4 &= A_3 + \frac{1}{2^3} + 1 \\ &= \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + 1 \end{aligned} \qquad \begin{aligned} A_5 &= A_4 - \frac{1}{2^4} - 1 \\ &= \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} \end{aligned}$$

Remark: At this point, especially if you are uncomfortable with logarithms, you may want to verify directly that $a_2 = 2^{\frac{1}{2}+1}$, $a_3 = 2^{\frac{1}{2}-\frac{1}{2^2}}$, $a_4 = 2^{\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}+1}$, and $a_5 = 2^{\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}}$.

From the emerging pattern, we guess that

$$A_{2r+1} = \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - + \cdots + \frac{1}{2^{2r-1}} - \frac{1}{2^{2r}}$$

for all $r \geq 0$ and that

$$A_{2r} = 1 + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - + \cdots - \frac{1}{2^{2r-2}} + \frac{1}{2^{2r-1}}$$

for all $r \geq 1$.

Using a trick for finding the sum of a *geometric series*, if we fix $r \geq 1$ and set

$$X = \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - + \cdots - \frac{1}{2^{2r-2}} + \frac{1}{2^{2r-1}}$$

then we have

$$2X = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + - \cdots - \frac{1}{2^{2r-3}} + \frac{1}{2^{2r-2}}.$$

When the two equations above are added, most of the terms cancel and we are left with $3X = 1 + \frac{1}{2^{2r-1}}$ or $X = \frac{1}{3}\left(1 + \frac{1}{2^{2r-1}}\right)$.

We now use this formula to refine our guesses to

$$\begin{aligned} A_{2r+1} &= X - \frac{1}{2^{2r}} \\ &= \frac{1}{3}\left(1 + \frac{1}{2^{2r-1}}\right) - \frac{1}{2^{2r}} \\ &= \frac{1}{3}\left(1 + \frac{2}{2^{2r}} - \frac{3}{2^{2r}}\right) \\ &= \frac{1}{3}\left(1 - \frac{1}{2^{2r}}\right) \end{aligned}$$

for $r \geq 0$, and similarly

$$\begin{aligned} A_{2r} &= 1 + X \\ &= 1 + \frac{1}{3}\left(1 + \frac{1}{2^{2r-1}}\right) \\ &= \frac{1}{3}\left(4 + \frac{1}{2^{2r-1}}\right) \end{aligned}$$

and these two guesses can be combined to get

$$A_n = \begin{cases} \frac{1}{3}\left(1 - \frac{1}{2^{n-1}}\right) & \text{if } n \text{ is odd} \\ \\ \frac{1}{3}\left(4 + \frac{1}{2^{n-1}}\right) & \text{if } n \text{ is even} \end{cases} \tag{4}$$

It is easily checked that Equation (4) holds for $n = 1$, $n = 2$, $n = 3$, and $n = 4$. As in part (a), we will use induction to prove that Equation (4) holds for all positive integers n.

Assume that Equation (4) holds for some positive integer k. If k is odd, Equation (4) means that $A_k = \frac{1}{3}\left(1 - \frac{1}{2^{k-1}}\right)$. Using Equation (3), we get

$$\begin{aligned}
A_{k+1} &= A_k + \frac{1}{2^k} + 1 \\
&= \frac{1}{3}\left(1 - \frac{1}{2^{k-1}}\right) + \frac{1}{2^k} + 1 \\
&= \frac{1}{3}\left(1 - \frac{1}{2^{k-1}} + \frac{3}{2^k} + 3\right) \\
&= \frac{1}{3}\left(4 - \frac{2}{2^k} + \frac{3}{2^k}\right) \\
&= \frac{1}{3}\left(4 + \frac{1}{2^k}\right)
\end{aligned}$$

which confirms that Equation (4) holds for the even integer $k+1$. If k is even, a similar calculation shows that Equation (4) holds for $k+1$. By mathematical induction, Equation (4) holds for all positive integers n.

Since $A_n = \log_2(a_n)$, we have $2^{A_n} = a_n$. Therefore,

$$a_n = \begin{cases} 2^{\frac{1}{3}\left(1 - \frac{1}{2^{n-1}}\right)} & \text{if } n \text{ is odd} \\ 2^{\frac{1}{3}\left(4 + \frac{1}{2^{n-1}}\right)} & \text{if } n \text{ is even} \end{cases} \tag{5}$$

(ii) For large values of n, the quantity $\dfrac{1}{2^{n-1}}$ gets very close to 0. Thus, for large odd values of n, a_n gets very close to $2^{\frac{1}{3}} = \sqrt[3]{2}$. For large even values of n, a_n gets very close to $2^{\frac{4}{3}} = 2\sqrt[3]{2}$.

Notice that $a_1, a_3, a_5, \ldots$ is a sequence of y-intercepts. As observed above, this sequence approaches the quantity $\sqrt[3]{2}$. Similarly, the sequence $a_2, a_4, a_6, \ldots$ is a sequence of x-intercepts and it approaches $2\sqrt[3]{2}$. The line through the points $(0, \sqrt[3]{2})$ and $(2\sqrt[3]{2}, 0)$ has slope $-\frac{1}{2}$. For large n, $\dfrac{1}{2^n}$ is close to 0, so m_n is close to $-\frac{1}{2^1} = -\frac{1}{2}$, which is exactly the slope computed in the previous sentence. You might want to think about why these values are the same.

(c) We will generalize the idea from part (b) above. For now, fix a positive number $b \neq 1$ and a real number c and set $m_n = -\dfrac{1}{b^{\frac{1}{2^n}+c}}$. By defining $A_n = \log_b(a_n)$, an almost identical calculation to the one in part (b) shows that

$$a_n = \begin{cases} b^{\frac{1}{3}\left(1 - \frac{1}{2^{n-1}}\right)} & \text{if } n \text{ is odd} \\ b^{\frac{1}{3}\left(3c + 1 + \frac{1}{2^{n-1}}\right)} & \text{if } n \text{ is even} \end{cases} \tag{6}$$

for every positive integer n.

As n goes to infinity, $\dfrac{1}{2^{n-1}}$ goes to 0, so the sequence $a_1, a_3, a_5, a_7, \ldots$ approaches $b^{\frac{1}{3}} = \sqrt[3]{b}$ and the sequence $a_2, a_4, a_6, a_8, \ldots$ approaches $b^{c+\frac{1}{3}}$. Thus, to answer this question, we can solve the system of equations

$$u = \sqrt[3]{b}$$
$$v = b^{c+\frac{1}{3}}$$

The first equation implies $b = u^3$. Notice that since $u \neq 1$, we indeed have $b \neq 1$, so the construction above will work. Substituting $b = u^3$ into the second equation gives $v = u^{3c+1}$. Taking $\log_u$ of both sides (which can be done since u and v are both positive) and solving gives $c = \frac{1}{3}(\log_u(v) - 1)$. Thus, we get the desired result by taking

$$m_n = -\frac{1}{b^{\frac{1}{2^n}+c}}$$

with $b = u^3$ and $c = \frac{1}{3}(\log_u(v) - 1)$.

(d) Using Equation (1), we can compute the following first few values of a_n.

$$a_1 = 1 \qquad a_2 = 1 \qquad a_3 = \frac{1}{2}$$
$$a_4 = \frac{1 \times 3}{2} \qquad a_5 = \frac{1 \times 3}{2 \times 4} \qquad a_6 = \frac{1 \times 3 \times 5}{2 \times 4}$$
$$a_7 = \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \qquad a_8 = \frac{1 \times 3 \times 5 \times 7}{2 \times 4 \times 6} \qquad a_9 = \frac{1 \times 3 \times 5 \times 7}{2 \times 4 \times 6 \times 8}$$

(i) Consider the following:

$$\begin{aligned}(a_5)^2 &= \frac{1 \times 3 \times 1 \times 3}{2 \times 4 \times 2 \times 4} \\ &= \frac{1 \times 3}{2^2} \times \frac{3}{4^2} \\ &= \frac{3}{4} \times \frac{3}{4^2} \\ &< \frac{3}{4^2}\end{aligned}$$

We can do similar calculations with a_7 and a_9 to get

$$\begin{aligned}(a_7)^2 &= \frac{1 \times 3 \times 5 \times 1 \times 3 \times 5}{2^2 \times 4^2 \times 6^2} \\ &= \frac{1 \times 3}{2^2} \times \frac{3 \times 5}{4^2} \times \frac{5}{6^2} \\ &= \frac{3}{4} \times \frac{15}{16} \times \frac{5}{6^2} \\ &< \frac{5}{6^2}\end{aligned}$$

$$\begin{aligned}(a_9)^2 &= \frac{1 \times 3}{2^2} \times \frac{3 \times 5}{4^2} \times \frac{5 \times 7}{6^2} \times \frac{7}{8^2} \\ &= \frac{3}{4} \times \frac{15}{16} \times \frac{35}{36} \times \frac{7}{8^2} \\ &< \frac{7}{8^2}\end{aligned}$$

Which shows that for $k = 2$, $k = 3$, and $k = 4$, we have

$$(a_{2k+1})^2 < \frac{2k-1}{(2k)^2} \tag{7}$$

By two applications of Equation (1), we get that $a_{2k+3} = a_{2k+1} \times \dfrac{2k+1}{2k+2}$ for every k. If we assume that Inequality (7) holds for some positive integer $k \geq 2$, we have

$$\begin{aligned}
(a_{2k+3})^2 &= (a_{2k+1})^2 \frac{(2k+1)^2}{(2k+2)^2} \\
&< \frac{2k-1}{(2k)^2} \times \frac{(2k+1)^2}{(2k+2)^2} && \text{(Inequality (7))} \\
&= \frac{(2k-1)(2k+1)}{(2k)^2} \times \frac{2k+1}{(2k+2)^2} \\
&= \frac{(2k)^2-1}{(2k)^2} \times \frac{2(k+1)-1}{(2(k+1))^2} \\
&< \frac{2(k+1)-1}{(2(k+1))^2}
\end{aligned}$$

which says that Inequality (7) holds for $k+1$. Since the odd integers are exactly those of the form $2k+1$ for some integer k, we have shown that $(a_n)^2 < \dfrac{n-2}{(n-1)^2}$ when $n \geq 3$ is odd. Since $n-2 < n-1$, we can simplify further to get that get that $(a_n)^2 < \dfrac{n-1}{(n-1)^2} = \dfrac{1}{n-1}$. It follows that $a_n < \dfrac{1}{\sqrt{n-1}}$ when $n \geq 3$ is odd. Setting $n = 10001$, we get that

$$a_{10001} < \frac{1}{\sqrt{10001-1}} = \frac{1}{\sqrt{10000}} = \frac{1}{100}$$

Note that $n = 10001$ is not the smallest n for which $a_n < \dfrac{1}{100}$. You may want to write a computer program to find the very first n for which $a_n < \dfrac{1}{100}$.

The inequality $a_n < \dfrac{1}{\sqrt{n-1}}$ for odd n shows us that the sequence $a_1, a_3, a_5, a_7, \ldots$ is approaching 0. This is because the quantity $\sqrt{n-1}$ goes to infinity as n goes to infinity, so its reciprocal goes to 0. The term a_n is between 0 and something that is getting closer and closer to 0, so it must also go to 0.

(ii) In much the same way as part (i), it can be shown that for even n, we have

$$\begin{aligned}
(a_n)^2 &= \frac{3^2}{2\times 4} \times \frac{5^2}{4\times 6} \times \frac{7^2}{6\times 8} \times \cdots \times \frac{(n-3)^2}{(n-4)\times(n-2)} \times \frac{(n-1)^2}{2\times(n-2)} \\
&> \frac{(n-1)^2}{2\times(n-2)} \\
&> \frac{(n-2)^2}{2(n-2)}
\end{aligned}$$

which implies $a_n > \dfrac{\sqrt{n-2}}{\sqrt{2}}$ when n is even. If we take $n = 20002$, we get

$$a_{20002} > \frac{\sqrt{20002-2}}{\sqrt{2}} = \sqrt{10000} = 100$$

again, $n = 20002$ is not the smallest n for which $a_n > 100$.