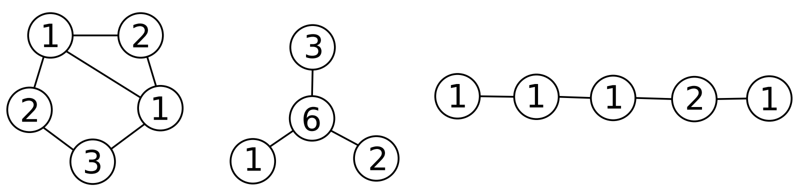
Problem of the Month

Additional Information about Problem 5: February 2023

Splendid sequences are part of a much more general curious combinatorial context. Here are a few definitions:

1. A *graph* is a collection of vertices (denoted in this document by circles) and edges (lines connecting some of these circles).
2. Two vertices in a graph connected by an edge are called *adjacent*.
3. A *splendid numbering* of a graph is an assignment of a positive integer to each vertex so that
   - Each number divides the sum of the integers in the adjacent vertices, and
   - the greatest common divisor of all integers is 1.

Below are some examples of splendid numberings of some graphs:

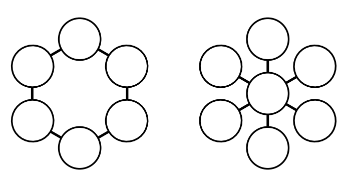


You may notice that a splendid sequence of length n is simply a splendid numbering of the graph that is a *path of length* n (the rightmost example above is a path of length 5).

So, given a graph, we can ask the same question as in the problem of the month: How many splendid numberings are there? Just like splendid sequences, the number of splendid numberings for any fixed graph is finite! (see part (e) of the problem).

The exact number of splendid numberings is known for paths of length n. This was essentially the content of part (f) of the problem.

Two other families of graph for which splendid numbers have been studied are n *cycles* and n-*pointed stars*. Rather than defining these generally, the image below has a 6 *cycle* on the left and a 6 *pointed star* on the right. We leave it to the reader to guess the general definition.



A 6 cycle (left) and a 6-pointed star (right)

The number of splendid numberings of an n cycle is known to be $\binom{2n-1}{n-1}$. This was computed in a paper from 2018 (see reference (1)), which is quite recent!

For an n-pointed star, the number of splendid numberings is closely related to the number of ways to decompose the number 1 as a sum of the form

$$1 = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$$

where each k_i is a positive integer, which is a very hard problem. In fact, it is currently unknown how many splendid numberings there are for the 9-pointed star.

This is almost all of what is known at the moment, and counting the number of splendid numberings (called arithmetical structures in the research community) is an active area of research. In 2020, for example, the number of splendid numberings for a particular family of graphs called "bidents" was computed (see reference (2)). It created a new entry in The On-Line Encyclopedia of Integer Sequences (OEIS)! The new entry is entry number A335676. If you've never played around with the OEIS, it's definitely worth it. Just make sure you have a few procrastination hours to burn.

As a final note, we should mention that there are many resources out there about Catalan numbers. For example, Richard P. Stanley's 2015 book "Catalan Numbers" contains a very long list of contexts where the Catalan numbers arise.

References

(1) Benjamin Braun et al. "Counting arithmetical structures on paths and cycles". In: *Discrete Math.* 341.10 (2018), pp. 2949-2963

(2) Kassie Archer et al. "Arithmetical structures on bidents". In: *Discrete Math.* 343.7 (2020), pp. 111850, 23