Problem of the Month

Problem 6: March 2023

For a non-negative integer n, define $f(n)$ to be the first digit after the decimal point in the decimal expansion of $\sqrt{n}$. For example, $\sqrt{10} = 3.162277\ldots$ and so $f(10) = 1$. Note that $f(0) = 0$ and that $f(n) = 0$ when n is a perfect square. You will likely want a calculator that can compute square roots for this question.

(a) Compute $f(n)$ for every integer n strictly between 1 and 4 as well as every integer n strictly between 36 and 49.

(b) Compute $f(n)$ for every integer n strictly between 4 and 9 as well as every integer n strictly between 49 and 64.

(c) Show that if n is a positive multiple of 5, then each digit from 0 through 9 appears in the list

$$f(n^2+1), f(n^2+2), f(n^2+3), \ldots, f(n^2+2n-1), f(n^2+2n)$$

the same number of times.

(d) For each digit d from 0 through 9, determine how many times d occurs in the list

$$f(1), f(2), f(3), \ldots, f(10^4)$$

(e) Here are a couple of other things that you might like to think about. No solution will be provided for either of these questions, but as always, we would love to hear about any observations you make!

- How are the digits 0 through 9 distributed among the infinite list

  $$f(1), f(2), f(3), f(4), \ldots$$

  For example, in the long run, are the ten digits distributed roughly "uniformly"? One way to make sense of this question is to think about the frequency of each digit among the list $f(1), f(2), f(3), \ldots, f(n)$ for very large n.

- Are there similar patterns to those in the earlier parts of this problem if we consider the first two digits after the decimal place? What if we consider three or more digits?