Problem of the Month

Problem 0: September 2023

In this problem, f will always be a function defined by $f(r) = \dfrac{ar+b}{cr+d}$ where a, b, c, and d are integers. These integers will vary throughout the parts of the problem.

Given such a function f and a rational number r_1, we can generate a sequence $r_1, r_2, r_3, \ldots$ by taking $r_n = f(r_{n-1})$ for each $n \geq 2$. That is, $r_2 = f(r_1)$, $r_3 = f(r_2)$, $r_4 = f(r_3)$, and so on. Unless there is some point in the sequence where $f(r_{n-1})$ is undefined, a sequence of this form can be made arbitrarily long.

These sequences behave in different ways depending on the function f and the starting value r_1. This problem explores some those behaviours.

(a) Suppose $f(r) = \dfrac{2r-1}{r+2}$.

- (i) With $r_1 = \frac{3}{2}$, compute r_2, r_3, and r_4.
- (ii) Find a rational number r_1 with the property that r_2 is defined, but r_3 is not defined.

(b) Suppose $f(r) = \dfrac{r+3}{2r-1}$.

- (i) With $r_1 = \frac{3}{7}$, compute r_2, r_3, r_4, and r_5.
- (ii) Determine all rational values of r_1 with the property that there is some integer $n \geq 1$ for which $f(r_n)$ is undefined. For all other values of r_1, find simplified formulas for r_{2023} and r_{2024} in terms of r_1.

(c) Suppose $f(r) = \dfrac{r+2}{r+1}$.

- (i) With $r_1 = 1$, compute r_2 through r_9. Write down decimal approximations of r_2 through r_9 (after computing them exactly).
- (ii) Suppose r is a positive rational number. Prove that

$$\left| \frac{f(r) - \sqrt{2}}{r - \sqrt{2}} \right| = \left| \frac{1 - \sqrt{2}}{r+1} \right|$$

- (iii) Suppose r_1 is a positive rational number. Prove that $\left| r_n - \sqrt{2} \right| < \dfrac{1}{2^{n-1}} \left| r_1 - \sqrt{2} \right|$ for each $n \geq 2$. Use this result to convince yourself that as n gets large, r_n gets close to $\sqrt{2}$, regardless of the choice of the positive value r_1. Can you modify f slightly so that the sequence always approaches $\sqrt{3}$?

(d) Explore the behaviour of the sequences generated by various values of r_1 for each of the functions below. Detailed solutions will not be provided, but a brief discussion will.

$$f(r) = \frac{r-3}{r-2}, \quad f(r) = \frac{r-1}{5r+3}, \quad f(r) = \frac{r-1}{r+2}, \quad f(r) = \frac{2r+2}{3r+3}, \quad f(r) = \frac{r+1}{r-2}$$