Problem of the Month

Problem 0: September 2021

Some friends are playing a game involving ten cards numbered 1 through 10. In part (a), Adina, Budi, and Dewei are the players. In parts (b) and (c), Adina, Budi, Charlie, and Dewei are the players. To play the game, each player other than Dewei chooses a card and shows it to all other players, but no player looks at their own card. The game consists of a dialogue with the goal being for all players holding a card to deduce the integer on their own card. In each part of this question, the dialogue is given in the order the statements/questions occurred. **No player is allowed ask a question to which they already know the answer**.

(a) Given the dialogue below, determine the integers on Adina's and Budi's cards.

1. (Adina) Is the integer on my card larger than the integer on Budi's card?
2. (Dewei) No.
3. (Budi) I know the integer on my card.
4. (Adina) I know the integer on my card.

(b) After the dialogue below, Adina, Budi, and Charlie each know the integer on their own card. Determine all possibilities for the integers on their cards.

1. (Adina) Is the sum of the integers on the cards a perfect square?
2. (Dewei) Yes.

(c) Given the dialogue below, determine all possibilities for the integers on the cards.

1. (Adina) Are the integers on any of the cards prime?
2. (Dewei) No.
3. (Budi) Is the sum of the integers on the cards prime?
4. (Dewei) Yes.
5. The three statements below occur simultaneously.
   - (Adina) I do not know what integer is on my card.
   - (Budi) I do not know what integer is on my card.
   - (Charlie) I know what integer is on my card.
6. The two statements below occur simultaneously
   - (Adina) I still do not know what integer is on my card.
   - (Budi) I now know what integer is on my card.
7. (Adina) I now know what integer is on my card.

Problem of the Month
Problem 1: October 2021

Suppose a, b, and c are positive integers. In this problem, a *non-negative solution* to the equation $ax + by = c$ is a pair $(x, y) = (u, v)$ of integers with $u \geq 0$ and $v \geq 0$ satisfying $au + bv = c$. For example, $(x, y) = (7, 0)$ and $(x, y) = (3, 3)$ are non-negative solutions to $3x + 4y = 21$, but $(x, y) = (-1, 6)$ is not.

(a) Determine all non-negative solutions to $5x + 8y = 120$.

(b) Determine the largest positive integer c with the property that there is no non-negative solution to $5x + 8y = c$.

In parts (c), (d), and (e), a and b are assumed to be positive integers satisfying $\gcd(a, b) = 1$.

(c) Determine the largest non-negative integer c with the property that there is no non-negative solution to $ax + by = c$. The value of c should be expressed in terms of a and b.

(d) Determine the number of non-negative integers c for which there are exactly 2021 non-negative solutions to $ax + by = c$. As with part (c), the answer should be expressed in terms of a and b.

(e) Suppose $n \geq 1$ is an integer. Determine the sum of all non-negative integers c for which there are exactly n nonnegative solutions to $ax + by = c$. The answer should be expressed in terms of a, b, and n.

Fact: You may find it useful that for integers a and b with $\gcd(a, b) = 1$, there always exist integers x and y such that $ax + by = 1$, though x and y may not be non-negative.

Problem of the Month

Problem 2: November 2021

A *lattice point* is a point (a, b) in the plane with the property that a and b are both integers. In this problem, we will say that a lattice point $P(a, b)$ is *visible* if $a > 0$, $b > 0$, and the line segment connecting P and the origin does not contain any lattice points other than P and the origin.

(a) How many lattice points $P(a, b)$ with $a \leq 10$ and $b \leq 10$ are visible?

(b) Determine the number of integers b with $b \leq 50$ for which $P(a, b)$ is visible when

(i) $a = 6$

(ii) $a = 18$

(ii) $a = 36$.

(c) Determine how many points $P(a, b)$ with $a \leq 50$ and $b \leq 50$ are visible. There is quite a bit to do by hand, so you may want to use technology to help.

(d) Explain why the following equality is true:

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\left(1 - \frac{1}{11^2}\right)\cdots \quad = \frac{1}{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots}$$

The expressions on the left is the infinite product of all expressions of the form $1 - \dfrac{1}{p^2}$ where p is prime.

(e) It is well known that the infinite sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

is equal to $\dfrac{\pi^2}{6}$. This fact has many proofs and is originally due to the mathematician Leonhard Euler. You may wish to explore some of these proofs, but the intention in this problem is for you to take the result for granted.

Interestingly, the probability that a randomly chosen point in the first quadrant not on the axes is visible is $\dfrac{6}{\pi^2}$. Explain why this is true.

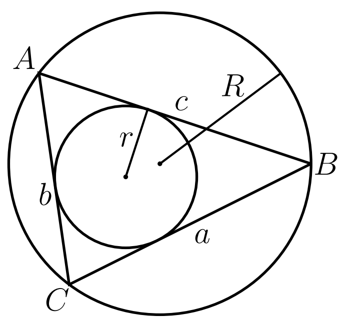
Note: It is ok to be a bit suspicious of what we mean by "probability" when choosing from an infinite set. Here is a way to think about what is meant in this problem: for a fixed positive integer, n, it is possible to compute the probability that a point $P(a, b)$ with $0 < a \leq n$ and $0 < b \leq n$ chosen randomly is visible. One might call this probability p_n. The question in (e), posed a bit more formally, might be "show that p_n gets very close to $\dfrac{6}{\pi^2}$ as n gets large". If you have seen *limits*, you might want to formalize this further.

Problem of the Month
Problem 3: December 2021

Before stating the problem, we will introduce some notation and terminology.

- In $\triangle ABC$, we will denote the length of BC by a, the length of AC by b, and the length of AB by c.
- The *semiperimeter* of $\triangle ABC$ will be denoted by s and is equal to $\dfrac{a+b+c}{2}$.
- The *incircle* of $\triangle ABC$ is the unique circle that is tangent to all three sides of $\triangle ABC$. Its radius is called the *inradius* of $\triangle ABC$ and is denoted by r. An important fact about the incircle is that its centre is at the intersection of the three angle bisectors of the triangle.
- The *circumcircle* of $\triangle ABC$ is the unique circle on which all three of A, B, and C lie. Its radius is called the *circumradius* of $\triangle ABC$ and is denoted by R. An important fact about the circumcircle is that its centre is at the intersection of the perpendicular bisectors of the three sides of the triangle.

The diagram below illustrates some of the information above.



This problem is about right-angled triangles. Most of us are aware of the famous Pythagorean theorem, but there are other interesting properties only satisfied by right-angled triangles.

(a) Suppose $\triangle ABC$ is right-angled at C and that h is the length of the altitude from C to AB. Show that $\dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{1}{h^2}$.

(b) Suppose $\triangle ABC$ is right-angled. Show that $\cos^2\angle A+\cos^2\angle B+\cos^2\angle C=1$.

(c) Suppose $\triangle ABC$ is right-angled. Show that $s=r+2R$.

(d) Suppose $\triangle ABC$ satisfies $a^2+b^2=c^2$. Prove that $\angle C=90^\circ$.

(e) Suppose $\triangle ABC$ satisfies $\cos^2\angle A+\cos^2\angle B+\cos^2\angle C=1$. Prove that $\triangle ABC$ is right-angled.

(f) Suppose $\triangle ABC$ satisfies $s=r+2R$. Show that $\triangle ABC$ is right-angled. [A solution to this problem will likely require some general identities involving the inradius and circumradius. Some specific useful identities will be given in the hint.]

Problem of the Month

Problem 4: January 2022

The goal of this problem is to work through some techniques that can sometimes help find the roots of polynomials. The statements of some parts of this problem refer to *repeated roots*, which we will now define. Suppose r is a root of the polynomial $p(x)$, that is, $p(r) = 0$. You may already know that if $p(r) = 0$, then $(x - r)$ divides evenly into $p(x)$. We say that r is a repeated root of $p(x)$ if $(x - r)^2$ divides evenly into $p(x)$. For example, 1 is a repeated root of $x^2 - 2x + 1$ because $x^2 - 2x + 1 = (x - 1)^2$, and 2 is a repeated root of $x^4 - 5x^3 + 6x^2 + 4x - 8$ since $x^4 - 5x^3 + 6x^2 + 4x - 8 = (x - 2)^2(x^2 - x - 2)$.

(a) The polynomials $p(x) = 2x^2 - 1275x + 194292$ and $q(x) = x^2 - 635x + 96516$ have a root in common. Determine both roots of both polynomials without using the quadratic formula.

(b) Let $p(x) = x^3 + ax^2 + bx + c$ be a polynomial with a root r. Show that r is a repeated root of $p(x)$ if and only if r is a root of the polynomial $q(x) = 3x^2 + 2ax + b$.

You may recognize $q(x)$ as the *derivative* of $p(x)$. If you are familiar with derivatives, you might want to try to generalize this part.

(c) Suppose $p(x) = x^3 + bx + c$ has roots u, v, and w (which may not all be different). Express the quantity $(u - v)^2(v - w)^2(w - u)^2$ in terms of b and c. This quantity is known as the discriminant of $p(x)$, and this exercise shows that its value can be determined from the coefficients without knowing the roots. Explain how, without knowing any of the roots, it is possible to determine if a cubic of the form $x^3 + bx + c$ has a repeated root.

(d) Consider the polynomial $p(x) = x^3 + ax^2 + bx + c$. Show that the coefficient of x^2 in the polynomial $q(x) = p\left(x - \frac{a}{3}\right)$ is equal to 0. Explain how the roots of $p(x)$ can be found easily if the roots of $q(x)$ are known.

(e) Find all roots of the polynomial $p(x) = x^3 - 135x^2 + 5832x - 81648$.

Problem of the Month
Problem 5: February 2022

In each part of this problem, there is a hallway containing of K doors numbered consecutively from 1 to K that are all initially closed. To *toggle* a door means to open it if it is closed and to close it if it is open. We will also use the notation that for a positive integer n, $\tau(n)$ is equal to the number of positive integer factors of n. For example, $\tau(1) = 1$ since 1 has exactly one positive factor and for a prime number p, we always have $\tau(p) = 2$ since prime numbers have exactly two positive factors. For another example $\tau(10) = 4$ since it has four positive integer factors, 1, 2, 5, and 10.

(a) In this part, $K = 100$. 100 "steps" are performed as follows:

- In step 1, every door that is numbered with a multiple of 1 is toggled.
- In step 2, every door that is numbered with a multiple of 2 is toggled.
- In step 3, every door that is numbered with a multiple of 3 is toggled.

In step n, every door that is numbered with a multiple of n is toggled. After all 100 steps are performed, which doors are open?

(b) In this part, $K = 100$. As with part (a), 100 steps are performed with one step for each integer n from 1 through K. This time, in step n, each door that is numbered with a multiple of n is toggled n times. For example, in step 5, each door that is numbered with a multiple of 5 is to be toggled 5 times. After all 100 steps are performed, which doors are open?

(c) In this part, $K = 2^9 \times 3^4 \times 5^{13} \times 7^{12}$. As with parts (a) and (b), a step is performed for each positive integer n from 1 through K. In step n, every door that is numbered by a multiple of n is toggled $\tau(n)$ times. For example, in step 5, every door that is numbered by a multiple of 5 is toggled $\tau(5) = 2$ times.

After all K steps are performed, is the door numbered with K open or closed?

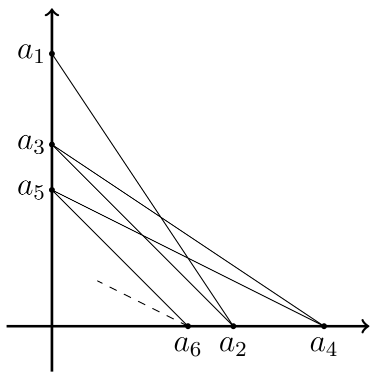
Problem of the Month

Problem 6: March 2022

In this problem, we will explore the following construction: Start with the positive real number $a_1 = 1$ and an infinite sequence $m_1, m_2, m_3, \ldots$ of negative slopes that are all distinct. For $n \geq 1$, we define a_{n+1} from a_n as follows.

- For odd n, a_{n+1} is the x-intercept of the line with slope m_n through $(0, a_n)$.
- For even n, a_{n+1} is the y-intercept of the line with slope m_n through $(a_n, 0)$.

The diagram below illustrates this. The line through $(0, a_1)$ and $(a_2, 0)$ has slope m_1, the line through $(a_2, 0)$ and $(0, a_3)$ has slope m_2, and so on.



(a) Suppose that $m_n = -\dfrac{1}{2^n}$ for all $n \geq 1$.

(i) Compute a_2, a_3, a_4, and a_5.

(ii) Find a general formula for a_n. You will likely need a separate formula for even n and odd n. Describe what happens to a_n as n gets large.

(b) Suppose that $m_n = -\dfrac{1}{2^{\frac{1}{2^n}+1}}$ for all n. [The exponent in the denominator is $\frac{1}{2^n} + 1$]

(i) Find a general formula for a_n.

(ii) Describe what happens to a_n as n gets large.

(c) Let u and v be arbitrary positive real numbers with $u \neq 1$. Give a sequence of slopes so that the sequence $a_1, a_3, a_5, a_7, \ldots$ approaches u and the sequence $a_2, a_4, a_6, a_8, \ldots$ approaches v. Remember that the sequence of slopes should not contain any repetitions.

(d) Suppose $m_n = -\dfrac{1}{n}$ for all $n \geq 1$.

(i) Find an integer n so that $a_n < \dfrac{1}{100}$.

(ii) Find an integer n so that $a_n > 100$.

Problem of the Month

Problem 7: April 2022

In this problem, a 3-*factorization* of a positive integer n is a triple (a, b, c) of positive integers such that $abc = n$. Two 3-factorizations will be regarded as the same if one of them can be obtained by reordering the integers in the other. For example, $(1, 2, 3)$ and $(2, 3, 1)$ are the same 3-factorization of 6. The *sum* of the 3-factorization (a, b, c) is $a + b + c$.

(a) Suppose n has two different 3-factorizations (a, b, c) and (d, e, f) with the same sum. Prove that at most one of these 3-factorizations contains the integer 1.

(b) Suppose n has two different 3-factorizations with the same sum. Prove that n has at least four prime factors. [We allow for repetition here. For instance, $24 = 2 \times 2 \times 2 \times 3$ has four prime factors, even though it only has two *distinct* prime factors.]

(c) Find the smallest integer n that has two different 3-factorizations with the same sum.

(d) Find an infinite family $n_1, n_2, n_3, \ldots$ of positive integers satisfying

- For each i, n_i has two different 3-factorizations with the same sum.
- For each i and j, $\gcd(n_i, n_j) = 1$.

(e) Challenge: Can you find an integer with three different 3-factorizations having the same sum? Can you find infinitely many such integers? Some direction on this will be given in the hint. You may wish to try to write a computer program to get started.

Problem of the Month

Problem 8: May 2022

In each part of this problem, a unit cube is positioned with its centre at the origin and is rotated about the x-axis so that it sweeps out a new "solid of revolution". To visualize this solid, you might imagine a cube being rotated very quickly about a fixed axis to produce an illusion of the solid. [This is the same phenomenon as when a rotating propellor or fan blade looks like a disk.] For example, if the cube is originally positioned so that the x-axis passes through the centres of two opposite faces, then the solid of revolution is a cylinder.

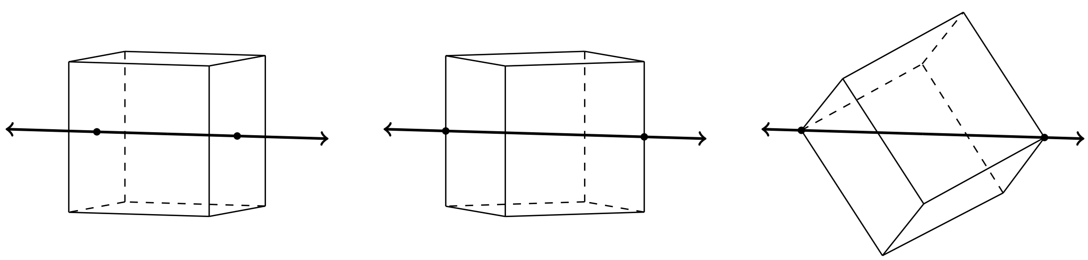
The solid of revolution depends on the original position of the cube. In each part, information is given to describe the original position of the cube and the goal is to describe the region in the (x, y)-plane intersected by the solid of revolution.

(a) The cube is positioned so that the x-axis passes through the centres of two opposite faces. As mentioned in the preamble, the solid is a cylinder.

(b) The cube is positioned so that the x-axis passes through the midpoints of two opposite edges of the cube (that is, two edges that are parallel and are not edges of the same face).

(c) The cube is positioned so that the x-axis passes through two opposite vertices of the cube (that is, two vertices that are not on a common face).

Below, from left to right, are diagrams of the original position of the cube for parts (a), (b), and (c), respectively. In order to avoid clutter in the diagrams, only the x-axis is included.



Notes:

- In the solutions, regions in the (x, y)-plane will have descriptions of the form "the region between $x = a$ and $x = b$ above the graph of $y = f(x)$ and below the graph of $y = g(x)$. You may have some other way of describing the regions.
- Solids of revolution are studied in calculus. If you already know some calculus and would like an added challenge, you might like to try to compute the area of the regions you find, or even the volumes of the solids of revolution.