The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Problem of the Week Problems and Solutions 2019-2020

Problem D (Grade 9/10)

Themes

Number Sense (N)
Geometry (G)
Algebra (A)
Data Management (D)
Computational Thinking (C)

(Click on a theme name above to jump to that section)

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*The problems in this booklet are organized into themes. A problem often appears in multiple themes.

Number Sense (N)

TAKE ME TO THE COVER

Problem of the Week
Problem D
An Average Report

A report card has six marks. The average of the first and second marks is 72%. The average of the second and third marks is 75%. The average of the third and fourth marks is 77%. The average of the fourth and fifth marks is 78%. The average of the fifth and sixth marks is 79%.

(a) Determine the overall average.

(b) Determine the average of the first and sixth marks.

Problem of the Week

Problem D and Solution

An Average Report

Problem

A report card has six marks. The average of the first and second marks is 72%. The average of the second and third marks is 75%. The average of the third and fourth marks is 77%. The average of the fourth and fifth marks is 78%. The average of the fifth and sixth marks is 79%. a) Determine the overall average. b) Determine the average of the first and sixth marks.

Solution

Let a, b, c, d, e, f represent the six report card marks.

The average of the first and second marks is 72, so $\frac{a+b}{2} = 72$. Multiplying by 2, $a + b = 144$. (1)

The average of the second and third marks is 75, so $\frac{b+c}{2} = 75$, leading to $b + c = 150$. (2)

The average of the third and fourth marks is 77, so $\frac{c+d}{2} = 77$, leading to $c + d = 154$. (3)

The average of the fourth and fifth marks is 78, so $\frac{d+e}{2} = 78$, leading to $d + e = 156$. (4)

The average of the fifth and sixth marks is 79, so $\frac{e+f}{2} = 79$, leading to $e + f = 158$. (5)

(a) To find the overall average we must find the sum $a + b + c + d + e + f$ and divide by 6. If we add equations (1), (3) and (5) we obtain the required sum.

$$\begin{aligned}(a+b)+(c+d)+(e+f) &= 144+154+158\\ a+b+c+d+e+f &= 456\\ \frac{a+b+c+d+e+f}{6} &= 76\end{aligned}$$

$\therefore$ The overall average is 76%.

(b) To find the average of the first and sixth marks, we must find the sum $a + f$ and divide by 2. We will add equations (1), (2), (3), (4) and (5).

$$\begin{aligned}(a+b)+(b+c)+(c+d)+(d+e)+(e+f) &= 144+150+154+156+158\\ a+2b+2c+2d+2e+f &= 762\\ a+2(b+c)+2(d+e)+f &= 762\\ a+2(150)+2(156)+f &= 762, \text{substituting from (2) and (4)}\\ a+300+312+f &= 762\\ a+f &= 150\\ \frac{a+f}{2} &= 75\end{aligned}$$

$\therefore$ the average of the first and sixth marks is 75%.

Problem of the Week
Problem D
It All Adds Up, Too

The *digit sum* of a positive integer is the sum of all of its digits.

For example, the *digit sum* of the integer 1234 is 10, since $1 + 2 + 3 + 4 = 10$.

Find all three-digit positive integers whose *digit sum* is exactly 9.

252

252

Problem of the Week
Problem D and Solution
It All Adds Up, Too

Problem

The *digit sum* of a positive integer is the sum of all of its digits. For example, the *digit sum* of the integer 1234 is 10, since $1 + 2 + 3 + 4 = 10$.
Find all three-digit positive integers whose *digit sum* is exactly 9.

Solution

There are twelve triples of digits that sum to 9. They are: (9,0,0), (8,1,0), (7,2,0), (7,1,1), (6,3,0), (6,2,1), (5,4,0), (5,3,1), (5,2,2) (4,4,1), (4,3,2), and (3,3,3).

We will count the number of three-digit positive integers whose *digit sum* is exactly 9 by counting the number of positive integers formed by the digits in each triple above. Notice that a triple can have two zeroes, one zero or no zeroes.

- Case 1: Two zeroes.

  The only triple with two zeros is (9,0,0). Since the first digit of the three-digit integer cannot be zero, the only three-digit integer made of these digits is 900. Therefore, there is only one integer in Case 1.

- Case 2: One zero.

  The triples with one zero are (8,1,0), (7,2,0), (6,3,0), and (5,4,0). Consider (8,1,0). Since the zero cannot be in the first position and can only be in the second or third positions, there are four integers that can be formed with these three digits: 810, 801, 180 and 108. Since this is true for any of the four triples in this case, there are $4 \times 4 = 16$ integers in Case 2.

- Case 3: No zeroes.

  There are three cases to consider: i) three repeated digits, ii) two repeated digits and iii) no repeated digits.

  i) Three repeated digits.

  The only triple with three repeated digits is (3,3,3) and the only integer formed by these digits is 333. Therefore, there is 1 integer in Case 3i).

ii) Two repeated digits.

The triples are (7,1,1), (5,2,2), and (4,4,1). Consider triple (7,1,1). There are three integers that can be formed by these three digits: 711, 171 and 117. Since this is true for any of the three triples in this case, there are $3 \times 3 = 9$ integers in Case 3 ii).

iii) No repeated digits.

The triples are (6,1,2), (5,3,1), and (4,3,2). Consider triple (6,2,1). There are six integers that can be formed by these three digits: 621, 612, 261, 216, 162 and 126. Since this is true for any of the three triples in this case, there are $3 \times 6 = 18$ integers in Case 3 iii).

Therefore, there are $1 + 9 + 18 = 28$ integers in Case 3.

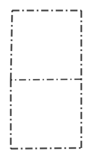
Therefore, the total number of three-digit positive integers that have a *digit sum* equal to 9 is $1 + 16 + 28 = 45$.

Note: It is a known fact that an integer is divisible by 9 exactly when its *digit sum* is divisible by 9. For example, 32 814 has a *digit sum* of $3 + 2 + 8 + 1 + 4 = 18$. Since 18 is divisible by 9, then 32 814 is divisible by 9. On the other hand, 32 810 has a *digit sum* of $3 + 2 + 8 + 1 + 0 = 14$. Since 14 is not divisible by 9, then 32 810 is not divisible by 9.

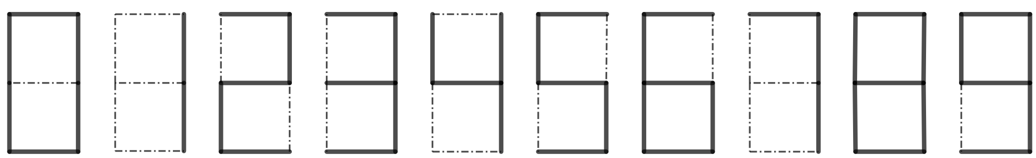
As a consequence of this fact, each of the 45 three-digit integers we found above must be divisible by 9.

Problem of the Week
Problem D
Seven Segments or Less

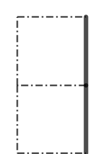
A 7-segment display consists of seven LEDs arranged in a rectangular fashion, as shown below. Each of the seven LEDs is called a segment because, when illuminated, the segment forms part of a numerical digit to be displayed. On a digital clock, each digit can be formed by lighting up some of the seven segments of the following 7-segment display.



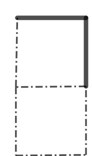
Below, each of the digits from 0 to 9 are shown by lighting up some of those seven segments. For example, the digit 8 uses all seven segments, but the digit 1 uses only the two right vertical segments.



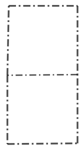
If a segment burns out, there could be a problem distinguishing which digit is showing. For example, if the top segment is burnt out then the display to the right could still be the digit 1 or it could be the digit 7.



However, if the bottom right vertical segment is the only segment burnt out, then we can unambiguously determine that the digit on the right must be the digit 7.



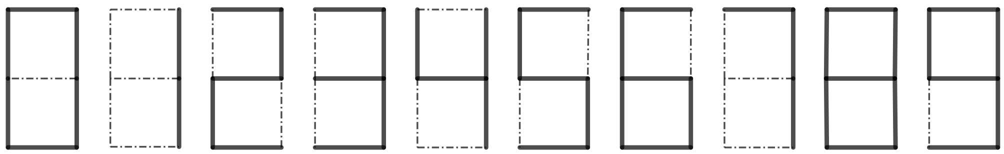
What is the fewest number of working segments that are needed so that each digit can be unambiguously determined?



Problem of the Week
Problem D and Solution
Seven Segments or Less

Problem

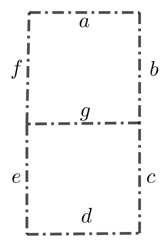
A 7-segment display consists of seven LEDs arranged in a rectangular fashion, as shown above. Each of the seven LEDs is called a segment because, when illuminated, the segment forms part of a numerical digit to be displayed. On a digital clock, each digit can be formed by lighting up some of the seven segments on a 7-segment display. Below, each of the digits from 0 to 9 are shown by lighting up some of those seven segments.



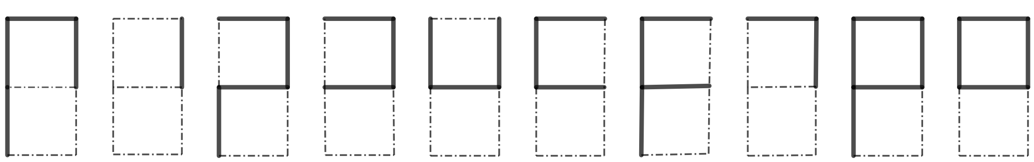
What is the fewest number of working segments that are needed so that each digit can be unambiguously determined?

Solution

Consider the following labelling of the 7 segments with the letters a through g:



Segment a must work to distinguish between digits 1 and 7.
Segment b must work to distinguish between digits 6 and 8.
Segment e must work to distinguish between digits 5 and 6 and also between the digits 8 and 9.
Segment f must work to distinguish between digits 3 and 9.
Segment g must work to distinguish between digits 0 and 8.
If we do not have c and d working, we would have the following 10 digits, each of which is unique:



Therefore, the fewest number of working segments needed is 5.

The Beaver Computing Challenge (BCC):

This problem is based on a previous BCC problem. The BCC is designed to get students with little or no previous experience excited about computing. Questions are inspired by topics in computer science and connections to Computer Science are described in the solutions to all past BCC problems. If you enjoyed this problem, you may want to explore the BCC contest further.

Connections to Computer Science:

Pattern recognition algorithms are algorithms with take in complex information, such as a picture or sound, and try to categorize it. For example, pattern recognition algorithms are used to detect facial features, such as eyes, mouth, and nose, for security and social media applications. These algorithms look for identifying information and attempt to provide a "most likely" answer, using statistical models.

Pattern recognition is a branch of a larger area of computer science called machine learning, that focuses on the recognition of patterns and regularities in data. One of the approaches to recognition is to extract specific features, that allow uniquely identified objects. This task focuses on identifying these key features to distinguish between each digit.

Problem of the Week

Problem D

The Number You Have Reached is ...

Maryam remembers a few things about a friend's cell phone number but cannot completely remember all ten digits. Maryam is absolutely certain that the first seven digits of the number are (122) 578 8.

Maryam does remember a few interesting things about the number. With regard to the numeric keypad, she remembers the following things:

- Any digit in the phone number that is different from the previous digit somehow touches the next digit of the phone number. For example, the digit 1 on the keypad touches digits 2, 4 and 5. The digit 5 on the keypad touches every digit but 0.
- The phone number contains three distinct pairs of repeating consecutive digits, but three digits in a row are never the same. (Of the numbers Maryam remembers, there are already two distinct pairs of repeating digits, so the third pair cannot be 22 or 88.)

1	2	3
4	5	6
7	8	9
*	0	#

Maryam's phone plan has unlimited free local calling. She will try different combinations until the friend is reached. How many different phone numbers could Maryam possibly end up trying until the correct number is reached?

Problem of the Week
Problem D and Solution
The Number You Have Reached is ...

Problem

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1	2	3
4	5	6
7	8	9
*	0	#

Maryam's phone plan has unlimited free local calling. She will try different combinations until the friend is reached. How many different phone numbers could Maryam possibly end up trying until the correct number is reached?

Solution

Six numbers on the keypad are adjacent to the 8. The first missing digit cannot be an 8 since there are never three consecutive digits the same. So, the only possibilities for the first missing digit are 4, 5, 6, 7, 9 and 0. We will examine each case.

1. **The first missing digit is a 4.**

   If the digit after the 4 is also a 4, we have the third pair of repeating digits. The final digit could be any of the five digits adjacent to the 4 on the keypad but it cannot be a 4. There are 5 possibilities for phone numbers in which the first two missing digits are 4's.

   If the digit after the 4 is not a 4, then the final two digits must be the same but cannot be 2's or 8's. There are then 3 possibilities for the final two digits, 11, 55 and 77.

   Therefore, if the first missing digit is a 4, then there are $5 + 3 = 8$ possible phone numbers.

2. **The first missing digit is a 5.**

   If the digit after the 5 is also a 5, we have the third pair of repeating digits. The final digit could be any of the eight digits adjacent to the 5 on the keypad but it cannot be a 5. There are 8 possibilities for phone numbers in which the first two missing digits are 5's.

   If the digit after the 5 is not a 5, then the final two digits must be the same but cannot be 2's or 8's. There are then 6 possibilities for the final two digits, 11, 33, 44, 66, 77 and 99.

   Therefore, if the first missing digit is a 5, then there are $8 + 6 = 14$ possible phone numbers.

3. The first missing digit is a 6.

If the digit after the 6 is also a 6, we have the third pair of repeating digits. The final digit could be any of the five digits adjacent to the 6 on the keypad but it cannot be a 6. There are 5 possibilities for phone numbers in which the first two missing digits are 6's.

If the digit after the 6 is not a 6, then the final two digits must be the same but cannot be 2's or 8's. There are then 3 possibilities for the final two digits, 33, 55 and 99.

Therefore, if the first missing digit is a 6, then there are $5 + 3 = 8$ possible phone numbers.

4. The first missing digit is a 7.

If the digit after the 7 is also a 7, we have the third pair of repeating digits. The final digit could be any of the four digits adjacent to the 7 on the keypad but it cannot be a 7. There are 4 possibilities for phone numbers in which the first two missing digits are 7's.

If the digit after the 7 is not a 7, then the final two digits must be the same but cannot be 2's or 8's. There are then 3 possibilities for the final two digits, 44, 55 and 00.

Therefore, if the first missing digit is a 7, then there are $4 + 3 = 7$ possible phone numbers.

5. The first missing digit is a 9.

If the digit after the 9 is also a 9, we have the third pair of repeating digits. The final digit could be any of the four digits adjacent to the 9 on the keypad but it cannot be a 9. There are 4 possibilities for phone numbers in which the first two missing digits are 9's.

If the digit after the 9 is not a 9, then the final two digits must be the same but cannot be 2's or 8's. There are then 3 possibilities for the final two digits, 55, 66 and 00.

Therefore, if the first missing digit is a 9, then there are $4 + 3 = 7$ possible phone numbers.

6. The first missing digit is a 0.

If the digit after the 0 is also a 0, we have the third pair of repeating digits. The final digit could be any of the three digits adjacent to the 0 on the keypad but it cannot be a 0. There are 3 possibilities for phone numbers in which the first two missing digits are zeroes.

If the digit after the 0 is not a 0, then the final two digits must be the same but cannot be 2's or 8's. There are then 2 possibilities for the final two digits, 77 and 99.

Therefore, if the first missing digit is a 0, then there are $3 + 2 = 5$ possible phone numbers.

By adding the number of possibilities from each distinct case we can determine the total number of possible phone numbers. There are $8 + 14 + 8 + 7 + 7 + 5 = 49$ possible phone numbers to try in order for Maryam to be able to get the friend's correct number.

Problem of the Week
Problem D
Thumbs Up for a Job Well Done

A team of employees completed a large project. Their manager gave them a big thumbs up. Later they were recognized more tangibly with a monetary bonus to share among themselves. The ages of the team members are consecutive integers and no one on the team has the same age. The oldest team member is 45.

The bonus was paid out as follows:

(i) \$1000 to the oldest member of the project team plus $\frac{1}{10}$ of what remains, then

(ii) \$2000 to the second oldest member of the project team plus $\frac{1}{10}$ of what then remains, then

(iii) \$3000 to the third oldest member of the project team plus $\frac{1}{10}$ of what then remains, and so on.

After all of the bonus money had been distributed, each member of the project team had received the same amount. What is the age of the youngest member of the project team?

Problem of the Week
Problem D and Solution
Thumbs Up for a Job Well Done

Problem

A team of employees completed a large project. Their manager gave them a big thumbs up. Later they were recognized more tangibly with a monetary bonus to share among themselves. The ages of the team members are consecutive integers and no one on the team has the same age. The oldest team member is 45.

The bonus was paid out as follows: \$1000 to the oldest member of the project team plus $\frac{1}{10}$ of what remains, then \$2000 to the second oldest member of the project team plus $\frac{1}{10}$ of what then remains, then \$3000 to the third oldest member of the project team plus $\frac{1}{10}$ of what then remains, and so on. After all of the bonus money had been distributed, each member of the project team had received the same amount. What is the age of the youngest member of the project team?

Solution

<u>Solution 1</u>

Let x represent the amount that each team member receives.
Let y represent the total amount of the bonus.
Then $y \div x$ is the number of team members.

The first team member gets \$1000 plus one-tenth of the remainder:

$$\begin{aligned}
& & x &= 1000 + \frac{1}{10}(y - 1000) \\
&\text{Multiply both sides by 10:} & 10x &= 10\ 000 + y - 1000 \\
&\text{Simplify and solve for } y: & y &= 10x - 9000 && (1)
\end{aligned}$$

The second team member gets \$2000 plus one-tenth of the remainder after the first team member's share and \$2000 is removed:

$$\begin{aligned}
& & x &= 2000 + \frac{1}{10}(y - x - 2000) \\
&\text{Multiply both sides by 10:} & 10x &= 20\,000 + y - x - 2000 \\
&\text{Simplify and solve for } y: & y &= 11x - 18\,000 && (2) \\
&\text{Since } y = y \text{ in (1) and (2):} & 10x - 9000 &= 11x - 18000 \\
& & \therefore x &= 9000 \\
&\text{Substitute for } x \text{ in (1):} & y &= 10(9000) - 9000 \\
& & &= 81\,000
\end{aligned}$$

Each team member receives \$9000 and the total bonus is \$81 000. The number of team members is $y \div x = 81\ 000 \div 9000 = 9$. The oldest team member is 45 and the ages of the team members are consecutive integers, so the youngest team member is 37.

Solution 2

Let n represent the number of team members.

Team member n (the youngest team member) receives $n \times \$1000$ or $1000n$. (1)

So $1000n$ is $\frac{9}{10}$ of what remains after team member $(n-1)$ is given $(n-1) \times 1000$.

$\therefore \frac{1}{10}$ of what remains is $\frac{1000n}{9}$.

Then team member $(n-1)$ receives $1000(n-1) + \frac{1000n}{9}$.

But team member $(n-1)$ and team member n each receive the same amount.

Therefore,

$$\begin{aligned} 1000(n-1) + \frac{1000n}{9} &= 1000n \\ 1000n - 1000 + \frac{1000n}{9} &= 1000n \\ \frac{1000n}{9} &= 1000 \\ 1000n &= 9000 \\ n &= 9 \\ \text{Substituting in (1):} \qquad 1000n &= 9\,000 \end{aligned}$$

There are 9 team members and each receives \$9 000. Since, the oldest team member is 45 and the ages of the team members are consecutive integers, the youngest team member is 37.

It is easy to think that the correct age of the youngest team member would be $45 - 9 = 36$. This actually turns out to be 10 people.

The ages of the group members are

37 38 39 40 41 42 43 44 45

There are 9 integers from 37 to 45, inclusive.

Problem of the Week

Problem D

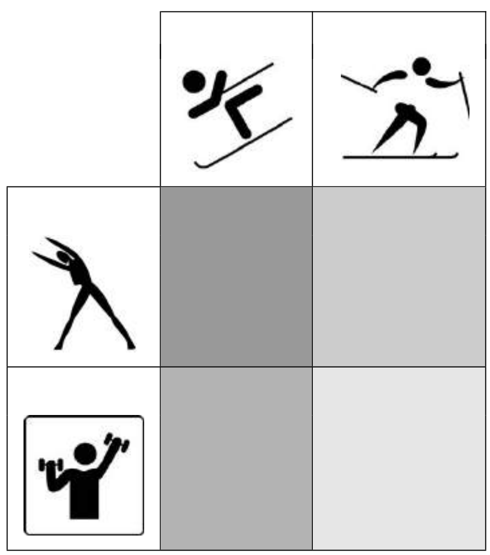
Ski Camp

For a winter ski camp, exactly 400 campers registered. Each camper registered for an outdoor activity, either downhill skiing or cross-country skiing. No camper could register for both of these outdoor activities. Also, each camper registered for an activity in the gym, either an aerobics group or a weight training group, but not both.

From the completed registrations, the following information is known:

- $\frac{5}{8}$ of the campers are registered for the aerobics group;
- 40% of the campers registered for downhill skiing are also registered for the aerobics group; and
- $66\frac{2}{5}\%$ of the campers registered for the aerobics group registered for cross-country skiing as their outdoor activity.

Determine the percentage of campers registered in both cross-country skiing and the weight training group.



Problem of the Week
Problem D and Solution
Ski Camp

Problem

For a winter ski camp, exactly 400 campers registered. Each camper registered for an outdoor activity, either downhill skiing or cross-country skiing. No camper could register for both of these outdoor activities. Also, each camper registered for an activity in the gym, either an aerobics group or a weight training group, but not both.

From the completed registrations, the following information is known: $\frac{5}{8}$ of the campers are registered for the aerobics group; 40% of the campers registered for downhill skiing are also registered for the aerobics group; and $66\frac{2}{5}\%$ of the campers registered for the aerobics group registered for cross-country skiing as their outdoor activity. Determine the percentage of campers registered in both cross-country skiing and the weight training group.

Solution

There were 400 students registered for the camp. Since $\frac{5}{8}$ registered for the aerobics group, $\frac{5}{8} \times 400$ or 250 campers were in the aerobics group. Since all 400 campers had to be in either the aerobics group or the weight training group, it follows that $400 - 250$ or 150 campers registered for the weight training group.

Since $66\frac{2}{5}\%$ of the campers in the aerobics group were registered for cross-country skiing, 0.664×250 or 166 campers were registered for both cross-country skiing and the aerobics group. The remainder of the aerobics group, $260 - 166$ or 84 campers, were in the aerobics group and downhill skiing.

But these 84 campers then represent 40% of the total number of campers registered for downhill skiing. Therefore, $84 \div 0.4$ or 210 campers are registered for downhill skiing. It follows that $210 - 84$ or 126 campers were registered in both weight training and downhill skiing.

The remaining campers had to be registered in both weight training and cross-country skiing. That is, $400 - 84 - 126 - 166 = 24$ campers were registered in both weight training and cross-country skiing. This number, expressed as a percentage, is $\frac{24}{400} \times 100\% = 6\%$.

Therefore, 6% of the campers registered for both cross-country skiing and weight training.

			TOTAL
	84	166	250
	126	24	150
	210	190	400

Problem of the Week
Problem D
This is the Year

The positive integers can be arranged as follows.

Row 1	1					
Row 2	2	3				
Row 3	4	5	6			
Row 4	7	8	9	10		
Row 5	11	12	13	14	15	
Row 6	16	17	18	19	20	21
$\vdots$						

More rows and columns continue to list the positive integers in order, with each new row containing one more integer than the previous row.

How many integers less than 2020 are in the *column* that contains the number 2020?

Did you know that the sum of the positive integers from 1 to n can be determined using the formula $\frac{n(n+1)}{2}$? That is, $1+2+3+4+\ldots+(n-1)+n=\frac{n(n+1)}{2}$. For example, the sum of the integers $1+2+3+4=\frac{4(5)}{2}=10$. This result can be verified by simply adding the 4 numbers. You can also easily verify that the sum of the first 5 positive integers is $\frac{5(6)}{2}=15$.

This formula may be useful in solving this problem. As an extension, one may wish to prove this formula holds for any positive integer n.

Problem of the Week
Problem D and Solution
This is the Year

Problem
The positive integers can be arranged as follows.

Row 1	1					
Row 2	2	3				
Row 3	4	5	6			
Row 4	7	8	9	10		
Row 5	11	12	13	14	15	
Row 6	16	17	18	19	20	21
⋮						

More rows and columns continue to list the positive integers in order, with each new row containing one more integer than the previous row. How many integers less than 2020 are in the *column* that contains the number 2020?

Solution

In the table given, there is one number in Row 1, there are two numbers in Row 2, three numbers in Row 3, and so on, with n numbers in Row n.
The numbers in the rows list the positive integers in order beginning at 1 in Row 1, with each new row containing one more integer than the previous row. Thus, the last number in each row is equal to the sum of the number of numbers in each row of the table up to that row.

For example, the last number in Row 4 is 10, which is equal to the sum of the number of numbers in rows 1, 2, 3, and 4. But the number of numbers in each row is equal to the row number. So 10 is equal to the sum $1 + 2 + 3 + 4$.
That is, the last number in Row n is equal to the sum

$$1 + 2 + 3 + 4 + \ldots + (n - 1) + n = \tfrac{n(n+1)}{2}.$$

To find out which row the integer 2020 occurs in, we could use organized trial and error.
Using trial and error, we find that since $\frac{63(63+1)}{2} = 2016$, then the last number in Row 63 is 2016.
We then find $\frac{64(64+1)}{2} = 2080$, so the last number in Row 64 is 2080.
Since 2020 is between 2016 and 2080, then it must appear somewhere in the 64th row.

Alternatively, we could determine the row the integer 2020 occurs in by using the

quadratic formula to solve the equation $\frac{n(n+1)}{2} = 2020$ for n.

$$\begin{aligned}
\frac{n(n+1)}{2} &= 2020 \\
n(n+1) &= 4040 \\
n^2 + n - 4040 &= 0 \\
n &\approx 63.1 \text{ (using the quadratic formula and } n \geq 0)
\end{aligned}$$

This means that the integer 2020 will be in Row 64.

If we look at the original arrangement given, the first number in Row 6, which is 16, has five numbers in the column above it. The second number in Row 6, which is 17, has four numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.

Also, note that the first number in Row 5, which is 11, has four numbers in the column above it. The second number in Row 5, which is 12, has three numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.

The pattern is that the first number in Row n has $n - 1$ numbers in the column above it. The second number in Row n has $n - 2$ numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.

Therefore, the first number of Row 64, which is 2017, will have 63 numbers in the column above it. The second number of Row 64, which is 2018, will have 62 numbers in the column above it. The third number of Row 64, which is 2019, will have 61 numbers in the column above it. The fourth number of Row 64, which is 2020, will have 60 numbers in the column above it.

Therefore, there are 60 integers less than 2020 in the column that contains the number 2020.

Problem of the Week

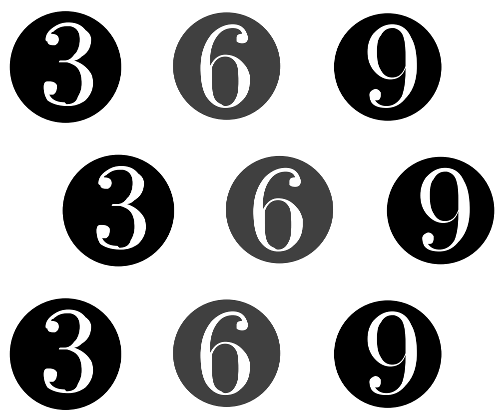
Problem D

Limiting the Possibilities

There are 9000 distinct 4-digit positive integers. If we add restrictions to the make up of the numbers, then the number of possible 4-digit numbers may be reduced.

How many 4-digit numbers can be created given the following restrictions?

- The only digits that can be used are 3, 6 and 9.
- A specific digit can occur at most three times in the number.
- The number must be divisible by 9.



Did You Know?

If the sum of the digits of a number is divisible by 9, then that number is divisible by 9. If the sum of the digits of a number is not divisible by 9, then that number is not divisible by 9.

For example, the sum of the digits of 3393 is 18, which is divisible by 9, so 3393 is divisible by 9. Also, the sum of the digits of 9933 is 24, which is not divisible by 9, so 9933 is not divisible by 9.

Problem of the Week
Problem D and Solution
Limiting the Possibilities

Problem

How many 4-digit numbers can be created such that the only digits that can be used are 3, 6 and 9, a specific digit can occur at most three times in the number, and the number must be divisible by 9?

Solution
Solution 1

This solution is provided for those who looked at every possible four-digit number that could be created from the given numbers. There is a number of ways to create the numbers and there would be 78 numbers to check. Of the 78 numbers, 26 of the numbers are divisible by 9. In a solution such as this, one must be careful not to miss any of the possibilities.

Solution 2

This solution looks at the possible sums that the digits can add to.

The minimum digit sum would be created using three 3s and one 6, yielding a sum of 15. The maximum digit sum would be created using three 9s and one 6, yielding a sum of 33. Neither of these sums is divisible by 9. However, between 15 and 33, there are two numbers, namely 18 and 27, which are divisible by 9. We are therefore looking for four digits whose sum is either 18 or 27. The following table looks at the possible digit combinations and the resulting sums.

Number of 3s	Number of 6s	Number of 9s	Digit Sum	Divisible by 9 (yes or no)
3	1	0	$3+3+3+6=15$	No
3	0	1	$3+3+3+9=18$	Yes
1	3	0	$3+6+6+6=21$	No
0	3	1	$6+6+6+9=27$	Yes
1	0	3	$3+9+9+9=30$	No
0	1	3	$6+9+9+9=33$	No
2	2	0	$3+3+6+6=18$	Yes
2	0	2	$3+3+9+9=24$	No
0	2	2	$6+6+9+9=30$	No
2	1	1	$3+3+6+9=21$	No
1	2	1	$3+6+6+9=24$	No
1	1	2	$3+6+9+9=27$	Yes

Now, we can create the numbers using the information in the rows of the chart in which the sum of the digits is divisible by 9.
First we can create all four-digit numbers having three 3s and one 9. The numbers are 3339, 3393, 3933 and 9333. There are 4 possible numbers.

Then we can create all four-digit numbers having three 6s and one 9. The numbers are 6669, 6696, 6966 and 9666. There are 4 possible numbers.

We continue by creating the four-digit numbers which contain two 3s and two 6s. The numbers are 3366, 3636, 3663, 6633, 6363, and 6336. There are 6 such numbers.

Finally, we create all numbers containing two 9s, one 3 and one 6. The numbers are 9936, 9963, 9396, 9693, 9369, 9639, 3996, 6993, 3969, 6939, 3699, and 6399. These numbers can be created systematically by placing the 9s in the six possible ways in a four-digit number. Then, for each of these ways of placing the 9s, the 3 and 6 can be filled in the remaining two spots in 2 ways. There are a total of 12 such numbers.

Combining the results, there are $4 + 4 + 6 + 12 = 26$ four-digit numbers, divisible by 9, that can be created using the given numbers.

Solution 3

We will consider four cases depending on the number of 9s in the four-digit number.

1. The number contains three 9s. Since the number contains three 9s, the digit sum is 27 plus the value of the fourth digit.
    - If the fourth digit is a 3, the digit sum is 30. Since 30 is not divisible by 9, no four-digit number containing three 9s and one 3 is divisible by 9.
    - If the fourth digit is a 6, the digit sum is 33. Since 33 is not divisible by 9, no four-digit number containing three 9s and one 6 is divisible by 9.

    This case produces no four-digit numbers which are divisible by 9.

2. The number contains two 9s. Since the number contains two 9s, the digit sum is 18 plus the sum of the two remaining digits.
    - If the two remaining digits are both 3s, the digit sum is 24. Since 24 is not divisible by 9, no four-digit number containing two 9s and two 3s is divisible by 9.
    - If the two remaining digits are both 6s, the digit sum is 30. Since 30 is not divisible by 9, no four-digit number containing two 9s and two 6s is divisible by 9.
    - If one of the remaining digits is a 3 and the other digit is a 6, the digit sum is 27. Since 27 is divisible by 9, four-digit numbers exist that contain two 9s, one 3 and one 6 which are divisible by 9. Using these digits we can create twelve four-digit numbers, each of which contain two 9s, one 3 and one 6. The numbers are 9936, 9963, 9396, 9693, 9369, 9639, 3996, 6993, 3969, 6939, 3699, and 6399. (These numbers can be found by putting the two 9s in the four-digit number in six different ways and then putting the 3 and 6 in each of these possibilities in two different ways.)

    This case produces 12 four-digit numbers which are divisible by 9.

3. The number contains one 9. Since the number contains one 9, the digit sum is 9 plus the sum of the three remaining digits.
   - If each of the three remaining digits is a 3, the digit sum is 18. Since 18 is divisible by 9, four-digit numbers exist that contain one 9 and three 3s which are divisible by 9. There are four possible numbers: 3339, 3393, 3933, and 9333.
   - If each of the three remaining digits is a 6, the digit sum is 27. Since 27 is divisible by 9, four-digit numbers exist that contain one 9 and three 6s which are divisible by 9. There are four possible numbers: 6669, 6696, 6966, and 9666.
   - If one of the remaining digits is a 3 and each of the other two remaining digits is a 6, the digit sum is 24. Since 24 is not divisible by 9, there are no four-digit numbers which contain one 9, one 3 and two 6s which are divisible by 9.
   - If one of the remaining digits is a 6 and each of the other two remaining digits is a 3, the digit sum is 21. Since 21 is not divisible by 9, there are no four-digit numbers which contain one 9, one 6 and two 3s which are divisible by 9.

   This case produces 8 four-digit numbers which are divisible by 9.

4. The number contains no 9s. Since the number contains no 9s, the digit sum is the sum of the four remaining digits.
   - If the number contains three 3s and one 6, the digit sum is 15. Since 15 is not divisible by 9, no four-digit number containing three 3s and one 6 is divisible by 9.
   - If the number contains three 6s and one 3, the digit sum is 21. Since 21 is not divisible by 9, no four-digit number containing three 6s and one 3 is divisible by 9.
   - If two of digits are 6s and two of the digits are 3s, the digit sum is 18. Since 18 is divisible by 9, four-digit numbers exist that contain two 6s and two 3s which are divisible by 9. There are 6 possible numbers: 3366, 3636, 3663, 6633, 6363, and 6336.

   This case produces 6 four-digit numbers which are divisible by 9.

Combining the results from the four cases, there are $0 + 12 + 8 + 6 = 26$ four-digit numbers, divisible by 9, that can be created using the given numbers.

Problem of the Week

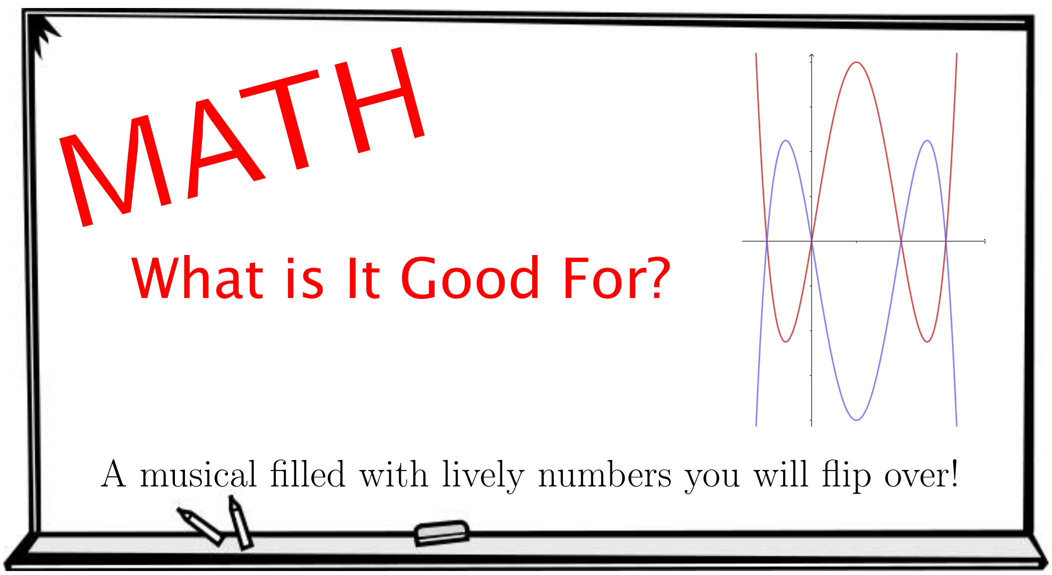
Problem D

Stick With It!

The junior and senior students at Mathville High School are going to present an exciting musical entitled, "Math, What is it Good For?". A large group of students came out to an information meeting. After a brief introduction to the musical, 15 senior students decided that it was not for them and they left. At that point, twice as many junior students as senior students remained.

Later in the meeting, after the 15 senior students had left, $\frac{3}{4}$ of the junior students and $\frac{1}{3}$ of the remaining senior students also left. This left 8 more senior students than junior students. All of the remaining students stuck it out and went on to produce an amazing product.

How many students remained to perform in the school musical, "Math, What is it Good For?"



MATH

Problem of the Week
Problem D and Solution
Stick With It!

What is It Good For?

Problem

The junior and senior students at Mathville High School are going to present an exciting musical entitled, "Math, What is it Good For?". A large group of students came out to an information meeting. After a brief introduction to the musical, 15 senior students decided that it was not for them and they left. At that point, twice as many junior students as senior students remained. Later in the meeting, after the 15 senior students had left, $\frac{3}{4}$ of the junior students and $\frac{1}{3}$ of the remaining senior students also left. This left 8 more senior students than junior students. All of the remaining students stuck it out and went on to produce an amazing product. How many students remained to perform in the school musical, "Math, What is it Good For?"

Solution

Let j represent the number of juniors and s represent the number of seniors present at the start of the information meeting.

After 15 seniors leave, $(s - 15)$ seniors remain. The number of juniors is twice the number of seniors at this point so $j = 2(s - 15)$ which simplifies to $j = 2s - 30$. (1)

Then $\frac{3}{4}$ of the juniors depart, leaving $\frac{1}{4}$ of the juniors or $\frac{1}{4}j$.
Also, $\frac{1}{3}$ of the seniors depart, leaving $\frac{2}{3}$ of the seniors remaining or $\frac{2}{3}(s - 15)$.
Now the number of seniors is 8 more than the number of juniors so $\frac{2}{3}(s - 15) = \frac{1}{4}j + 8$.
Multiplying by 12, the equation simplifies to $8(s - 15) = 3j + 96$. This further simplifies to $8s - 120 = 3j + 96$. (2)

At this point we can use substitution or elimination to solve the system of equations. In this case, we substitute (1) into (2) for j to solve for s.

$$\begin{aligned} 8s - 120 &= 3(2s - 30) + 96 \\ 8s - 120 &= 6s - 90 + 96 \\ 2s &= 126 \\ s &= 63 \end{aligned}$$

Substituting $s = 63$ into (1), $j = 2(63) - 30 = 126 - 30 = 96$.

The original number of students is $j + s = 96 + 63 = 159$.

The number of juniors remaining is $\frac{1}{4}j = \frac{1}{4}(96) = 24$.

The number of seniors remaining is $\frac{2}{3}(s - 15) = \frac{2}{3}(63 - 15) = \frac{2}{3}(48) = 32$.

The total number of students remaining to successfully perform in the musical is $24 + 32 = 56$.

The final production was an amazing success even though only 56 of the initial 159 students remained.

Problem of the Week

Problem D

Parting Ways

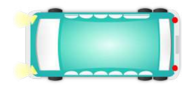
At 7:00 a.m., Sahil drives north at 48 km/h. At the same time from the same intersection, Brenda drives west at 64 km/h.

At what time will they be 260 km apart?

Problem of the Week

Problem D and Solution

Parting Ways



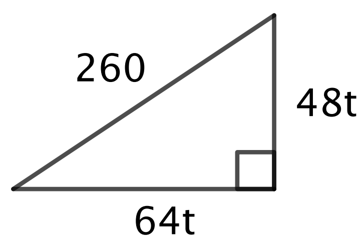
Problem

At 7:00 a.m., Sahil drives north at 48 km/h. At the same time from the same intersection, Brenda drives west at 64 km/h. At what time will they be 260 km apart?

Solution

Let t be the length of time, in hours, that Sahil and Brenda travel until they are 260 km apart. Since Sahil is travelling at 48 km/h, he will travel $48t$ km in t hours. Since Brenda is travelling at 64 km/h, she will travel $64t$ km in t hours.

Since Sahil is travelling north and Brenda is travelling west, they are travelling at right angles to each other. We can represent the distances on the following right triangle.



Using the Pythagorean Theorem

$$\begin{aligned} (48t)^2 + (64t)^2 &= 260^2 \\ 2304t^2 + 4096t^2 &= 67600 \\ 6400t^2 &= 67600 \\ 16t^2 &= 169 \\ t^2 &= \frac{169}{16} \end{aligned}$$

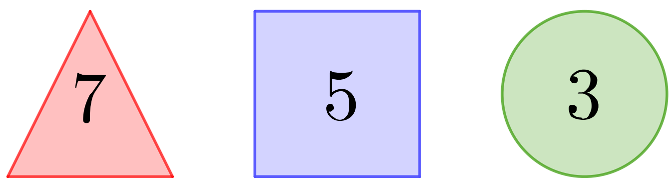
Since $t > 0, t = \frac{13}{4} = 3.25$, which is equivalent to 3 hours and 15 minutes. Note, 3 h 15 min after 7:00 a.m. is 10:15 a.m. Also, $48t = 48 \times \frac{13}{4} = 156$ and $64t = 64 \times \frac{13}{4} = 208$.

Therefore at 10:15 a.m. Sahil and Brenda are 260 km apart. Sahil has travelled 156 km and Brenda has travelled 208 km.

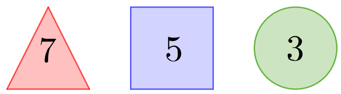
Problem of the Week
Problem D
Tokens Taken

Three bags each contain tokens. The green bag contains 22 round green tokens, each with a different integer from 1 to 22. The red bag contains 15 triangular red tokens, each with a different integer from 1 to 15. The blue bag contains 10 square blue tokens, each with a different integer from 1 to 10.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $22 \times 15 \times 10 = 3300$ different combinations of tokens created by selecting one token from each bag. Note that selecting the 7 red token, the 5 blue token and 3 green token is different than selecting the 5 red token, 7 blue token and the 3 green token. The order of selection does not matter.



You select one token from each bag. What is the probability that two or more of the selected tokens have the number 5 on them?



Problem of the Week
Problem D and Solution
Tokens Taken

Problem

Three bags each contain tokens. The green bag contains 22 round green tokens, each with a different integer from 1 to 22. The red bag contains 15 triangular red tokens, each with a different integer from 1 to 15. The blue bag contains 10 square blue tokens, each with a different integer from 1 to 10.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $22 \times 15 \times 10 = 3300$ different combinations of tokens created by selecting one token from each bag. Note that selecting the 7 red token, the 5 blue token and 3 green token is different than selecting the 5 red token, 7 blue token and the 3 green token. The order of selection does not matter.

You select one token from each bag. What is the probability that two or more of the selected tokens have the number 5 on them?

Solution

Solution 1

There are 22 different numbers which can be chosen from the green bag, 15 different numbers which can be chosen from the red bag, and 10 different numbers which can be chosen from the blue bag. So there are a total of $22 \times 15 \times 10 = 3300$ different combinations of numbers which can be produced by selecting one token from each bag.

To count the number of possibilities for a 5 to appear on at least two of the tokens, we will consider cases.

1. Each of the selected tokens has a 5 on it.

   This can only occur in 1 way.

2. A 5 appears on the green token and on the red token but not on the blue token.

   There are 9 choices for the blue token excluding the 5. A 5 can appear on the green token and on the red token but not on the blue token in 9 ways.

3. A 5 appears on the green token and on the blue token but not on the red token.

   There are 14 choices for the red token excluding the 5. A 5 can appear on the green token and on the blue token but not on the red token in 14 ways.

4. A 5 appears on the red token and on the blue token but not on the green token.

   There are 21 choices for the green token excluding the 5. A 5 can appear on the red token and on the blue token but not on the green token in 21 ways.

Summing the results from each of the cases, the total number of ways for a 5 to appear on at least two of the tokens is $1 + 9 + 14 + 21 = 45$. The probability of 5 appearing on at least two of the tokens is $\frac{45}{3300} = \frac{3}{220}$.

Solution 2

This solution uses a known result from probability theory. If the probability of event A occurring is a, the probability of event B occurring is b, the probability of event C occurring is c, and the results are not dependent on each other, then the probability of all three events happening is $a \times b \times c$.

The probability of a specific number being selected from the green bag is $\frac{1}{22}$ and the probability of any specific number not being selected from the green bag is $\frac{21}{22}$.

The probability of a specific number being selected from the red bag is $\frac{1}{15}$ and the probability of any specific number not being selected from the red bag is $\frac{14}{15}$.

The probability of a specific number being selected from the blue bag is $\frac{1}{10}$ and the probability of any specific number not being selected from the blue bag is $\frac{9}{10}$.

In the following we will use $P(p, q, r)$ to mean the probability of p being selected from the green bag, q being selected from the red bag, and r being selected from the blue bag. So, $P(5, 5, \text{ not } 5)$ means that we want the probability of a 5 being selected from the green bag, a 5 being selected from the red bag, and anything but a 5 being selected from the blue bag.

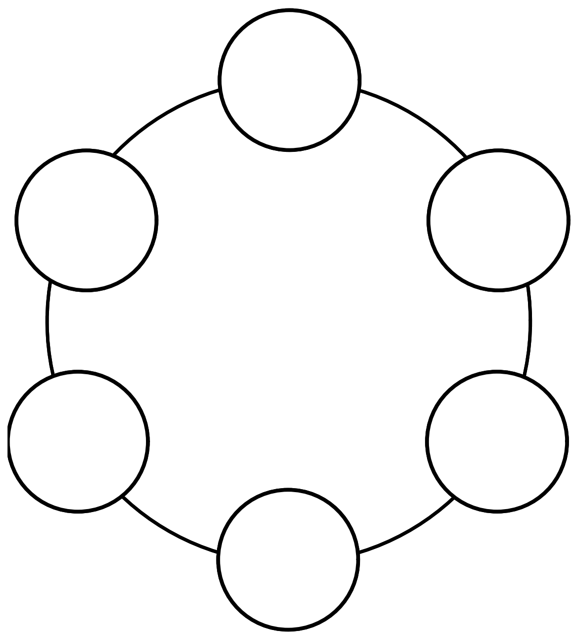
$$\begin{aligned}
& \text{Probability of 5 being selected from at least two of the bags} \\
= \; & \text{Probability of 5 from each bag} + \text{Probability of 5 from exactly 2 bags} \\
= \; & \text{P(5, 5, 5)} + \text{P(5, 5, not 5)} + \text{P(5, not 5, 5)} + \text{P(not 5, 5, 5)} \\
= \; & \frac{1}{22} \times \frac{1}{15} \times \frac{1}{10} + \frac{1}{22} \times \frac{1}{15} \times \frac{9}{10} + \frac{1}{22} \times \frac{14}{15} \times \frac{1}{10} + \frac{21}{22} \times \frac{1}{15} \times \frac{1}{10} \\
= \; & \frac{1}{3300} + \frac{9}{3300} + \frac{14}{3300} + \frac{21}{3300} \\
= \; & \frac{45}{3300} \\
= \; & \frac{3}{220}
\end{aligned}$$

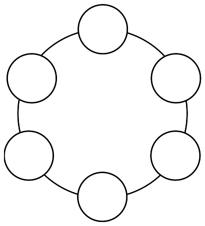
The probability of 5 appearing on at least two of the tokens is $\dfrac{3}{220}$.

Problem of the Week
Problem D
A Circle of Numbers

The numbers 1, 6, 8, 13, 15, and 20 can be placed in the circle below, each exactly once, so that the sum of each pair of numbers adjacent in the circle is a multiple of seven.

In fact, there is more than one way to arrange the numbers in such a way in the circle. Determine all different arrangements. Note that we will consider two arrangements to be the same if one can be obtained from the other by a series of reflections and rotations.





Problem of the Week
Problem D and Solution
A Circle of Numbers

Problem

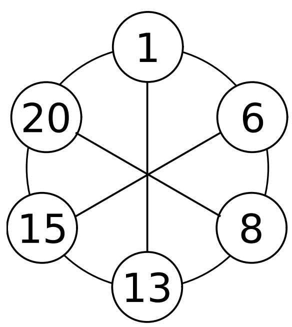
The numbers 1, 6, 8, 13, 15, and 20 can be placed in the circle above, each exactly once, so that the sum of each pair of numbers adjacent in the circle is a multiple of seven. In fact, there is more than one way to arrange the numbers in such a way in the circle. Determine all different arrangements. Note that we will consider two arrangements to be the same if one can be obtained from the other by a series of reflections and rotations.

Solution

We will start by writing down all the pairs of numbers that add to a multiple of 7.

Sum of 7	Sum of 14	Sum of 21	Sum of 28	Sum of 35
1,6	1,13	6,15	13,15	15,20
	6,8	8,13	8,20	
		1,20		

To show these connections visually, we can write the numbers in a circle and draw a line connecting numbers that add to a multiple of 7.

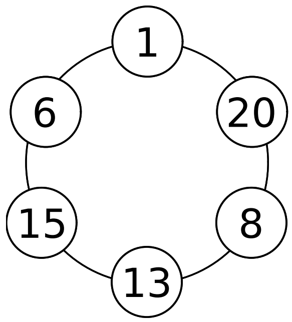


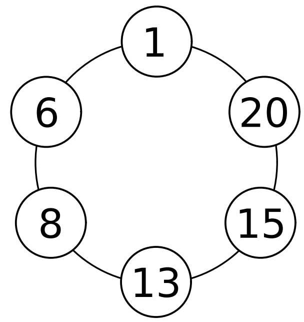
We will now determine all the different arrangements by looking at various cases. Note that in order for two arrangements to be different, at least some of the numbers need to be adjacent to different numbers.

Now, consider the possibilities for the numbers adjacent to 1. Since 6, 13, and 20 are the only numbers in our list that add with 1 to make a multiple of 7, there are three possible cases: 1 adjacent to 6 and 20, 1 adjacent to 6 and 13, and 1 adjacent to 13 and 20. We consider each case separately.

Case 1: 1 is adjacent to 6 and 20

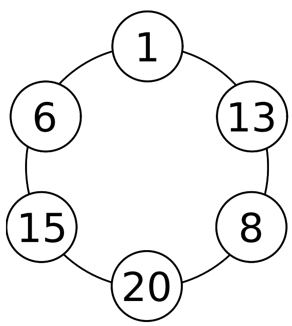
In this case, we can see from our table that 13 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.

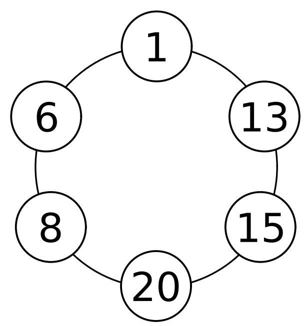




Case 2: 1 is adjacent to 6 and 13

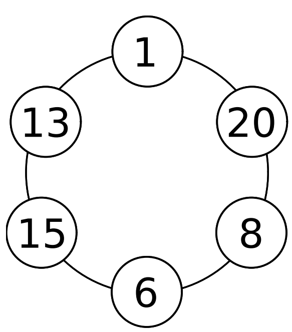
In this case, we can see from our table that 20 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.

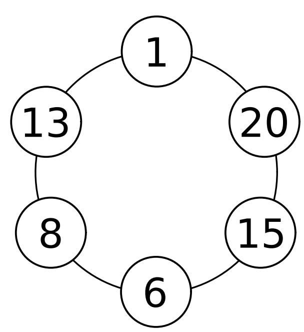




Case 3: 1 is adjacent to 13 and 20

In this case, we can see from our table that 6 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.





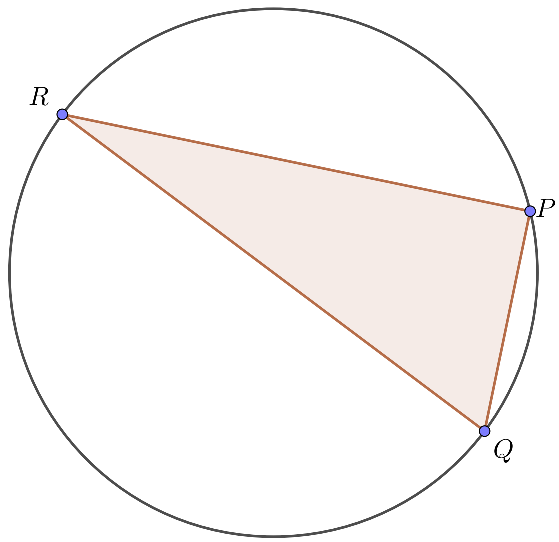
Therefore, we have found that there are 6 different arrangements. These are the arrangements shown in Cases 1, 2 and 3 above.

Geometry (G)

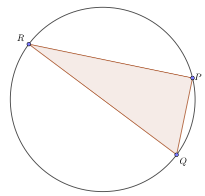
TAKE ME TO THE COVER

Problem of the Week
Problem D
What's Left?

A triangle is said to be inscribed in a circle when all three vertices of the triangle are on the circle. $\triangle PQR$ is inscribed in a circle with vertices Q and R located at the endpoints of a diameter of the circle and the third vertex, P, on the circumference of the circle so that $PR = 24$ cm, $PQ = 10$ cm, and $QR = 26$ cm.



Determine the area of the unshaded region of the circle. Express the area as an exact answer involving π. Then state the area correct to the nearest hundredth cm^2.



Problem of the Week
Problem D and Solution
What's Left?

Problem

A triangle is said to be inscribed in a circle when all three vertices of the triangle are on the circle. $\triangle PQR$ is inscribed in a circle with vertices Q and R located at the endpoints of a diameter of the circle and the third vertex, P, on the circumference of the circle so that $PR = 24$ cm, $PQ = 10$ cm, and $QR = 26$ cm. Determine the area of the unshaded region of the circle. Express the area as an exact answer involving π. Then state the area correct to the nearest hundredth cm^2.

Solution

In $\triangle PQR$, $PQ^2 + PR^2 = 10^2 + 24^2 = 100 + 576 = 676 = 26^2 = QR^2$.
Therefore, by the Pythagorean Theorem, $\triangle PQR$ is right-angled and $\angle P = 90°$.

We can use PQ as a base and PR as a height to determine the area of $\triangle PQR$.

$$\text{area } \triangle PQR = (PQ)(PR) \div 2 = (10)(24) \div 2 = 120 \text{ cm}^2$$

In the circle, QR is a diameter and $QR = 26$ cm. Therefore, the radius of the circle is 13 cm.

We can now determine the area of the circle.

$$\text{area of circle} = \pi r^2 = \pi(13)^2 = 169\pi \text{ cm}^2$$

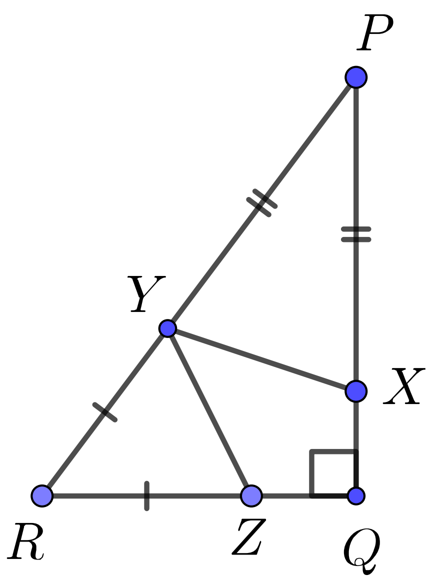
The unshaded area is calculated by subtracting the area of the triangle from the area of the circle.

$$\text{Uushaded area} = (169\pi - 120) \text{ cm}^2 \doteq 410.93 \text{ cm}^2$$

Therefore, the unshaded area is $(169\pi$ - $120)$ cm^2 or approximately 410.93 cm^2.

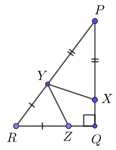
There is a known result for circles that could have been used in this problem. An angle inscribed in a semi-circle by the diameter of the circle is always 90°. This tells us that $\angle QPR = 90°$. With this fact, we could have omitted the use of the Pythagorean Theorem at the beginning of the solution.

Problem of the Week
Problem D
Angles Missing?



In right $\triangle PQR$, X is on PQ, Z is on QR and Y is on PR such that $PX = PY$ and $RZ = RY$.

Determine the measure of $\angle XYZ$.



Problem of the Week

Problem D and Solution

Angles Missing?

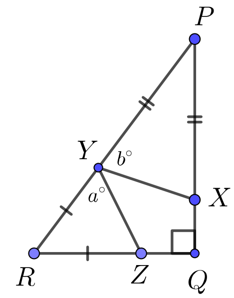
Problem

In right $\triangle PQR$, X is on PQ, Z is on QR and Y is on PR such that $PX = PY$ and $RZ = RY$.

Determine the measure of $\angle XYZ$.

Solution

Solution 1



Let $\angle RYZ = a^\circ$.

Since $RY = RZ$, $\triangle RYZ$ is isosceles and $\angle RZY = \angle RYZ = a^\circ$.

In a triangle, the angles sum to 180°.

$$\begin{aligned} \text{In } \triangle RYZ,\quad \angle R + \angle RYZ + \angle RZY &= 180^\circ \\ \angle R + a^\circ + a^\circ &= 180^\circ \\ \angle R &= 180^\circ - 2a^\circ \end{aligned}$$

$$\begin{aligned} \text{In } \triangle PQR,\quad \angle P + \angle Q + \angle R &= 180^\circ \\ \angle P + 90^\circ + (180^\circ - 2a^\circ) &= 180^\circ \\ \angle P &= 2a^\circ - 90^\circ \end{aligned}$$

Let $\angle PYX = b^\circ$.

Since $PY = PX$, $\triangle PYX$ is isosceles and $\angle PXY = \angle PYX = b^\circ$.

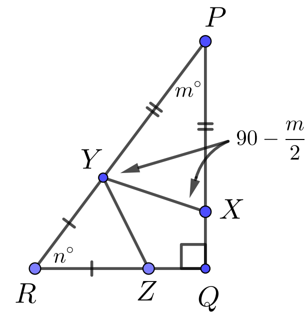
$$\begin{aligned} \text{In } \triangle PYX,\quad \angle PYX + \angle PXY + \angle P &= 180^\circ \\ b^\circ + b^\circ + (2a^\circ - 90^\circ) &= 180^\circ \\ 2b^\circ &= 270^\circ - 2a^\circ \\ b^\circ &= 135^\circ - a^\circ \end{aligned}$$

Now, PYR forms a straight line so $\angle PYR = 180^\circ$.

$$\begin{aligned} \text{So, } \angle PYX + \angle XYZ + \angle RYZ &= 180^\circ \\ b^\circ + \angle XYZ + a^\circ &= 180^\circ \\ (135^\circ - a^\circ) + \angle XYZ + a^\circ &= 180^\circ \\ \angle XYZ &= 180^\circ - 135^\circ \\ \therefore \angle XYZ &= 45^\circ \end{aligned}$$

Note: In solving for $\angle XYZ$ it is was not necessary to find either a° or b°.

Solution 2



In $\triangle PQR$, let $\angle P = m^\circ$ and $\angle R = n^\circ$.

$$\begin{aligned} \text{In } \triangle PQR, \quad \angle P + \angle Q + \angle R &= 180^\circ \\ m^\circ + 90^\circ + n^\circ &= 180^\circ \\ m^\circ + n^\circ &= 90^\circ \quad (1) \end{aligned}$$

Since $PY = PX$, $\triangle PYX$ is isosceles and therefore $\angle PXY = \angle PYX$. Furthermore, we will show $\angle PXY = \angle PYX = \left(90 - \frac{m}{2}\right)^\circ$.

$$\begin{aligned} \text{In } \triangle PYX, \quad \angle PYX + \angle PXY + \angle P &= 180^\circ \\ \angle PYX + \angle PYX + m^\circ &= 180^\circ \\ 2\angle PYX &= 180^\circ - m^\circ \\ \angle PYX &= \left(90 - \frac{m}{2}\right)^\circ \\ \text{and } \angle PXY &= \left(90 - \frac{m}{2}\right)^\circ \end{aligned}$$

Similarly, since $RY = RZ$, $\triangle RYZ$ is isosceles, and therefore $\angle RYZ = \angle RZY$. Furthermore, we will show $\angle RYZ = \angle RZY = \left(90 - \frac{n}{2}\right)^\circ$.

$$\begin{aligned} \text{In } \triangle RYZ, \quad \angle RYZ + \angle RZY + \angle R &= 180^\circ \\ \angle RYZ + \angle RYZ + n^\circ &= 180^\circ \\ 2\angle RYZ &= 180^\circ - n^\circ \\ \angle RYZ &= \left(90 - \frac{n}{2}\right)^\circ \\ \text{and } \angle RZY &= \left(90 - \frac{n}{2}\right)^\circ \end{aligned}$$

Now, PYR forms a straight line, so $\angle PYR = 180^\circ$.

$$\begin{aligned} \text{So, } \angle PYX + \angle XYZ + \angle RYZ &= 180^\circ \\ \left(90 - \frac{m}{2}\right)^\circ + \angle XYZ + \left(90 - \frac{n}{2}\right)^\circ &= 180^\circ \\ \left(90 - \frac{m}{2}\right)^\circ + \left(90 - \frac{n}{2}\right)^\circ + \angle XYZ &= 180^\circ \\ 180^\circ - \left(\frac{m}{2}\right)^\circ - \left(\frac{n}{2}\right)^\circ + \angle XYZ &= 180^\circ \\ \angle XYZ &= \left(\frac{m}{2}\right)^\circ + \left(\frac{n}{2}\right)^\circ \\ \angle XYZ &= \left(\frac{m+n}{2}\right)^\circ \\ \angle XYZ &= \left(\frac{90}{2}\right)^\circ \text{ (from (1))} \\ \therefore \angle XYZ &= 45^\circ \end{aligned}$$

Note: In solving for $\angle XYZ$ it is was not necessary to find either m° or n°.

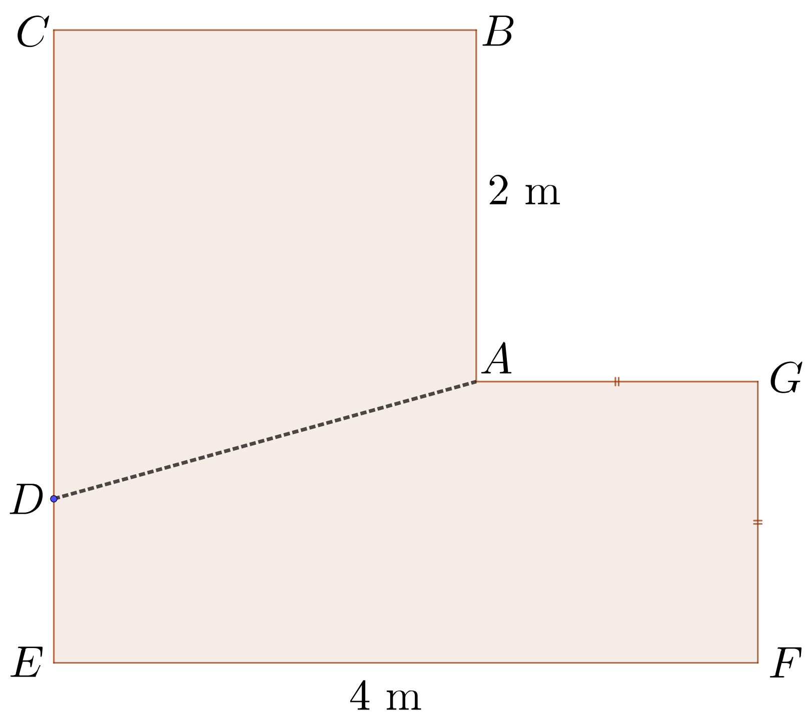
Problem of the Week

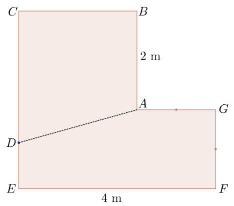
Problem D

It Must be Fair!

Two children share an L-shaped bedroom. They are constantly fighting over space. Their parents decide to temporarily partition the room with a curtain so that each child will have exactly the same area, in the hope that the space arguments will end.

The layout of the room is represented by $ABCDEFG$ on the diagram. The room has square corners with $EF = 4$ m, $AB = 2$ m, and $AG = GF$. The area of the entire room is 11.2 m^2. The partitioning curtain is to be hung from point A to a point D on CE to divide the room into two parts of equal area. Where is D located on CE to accomplish the equal area split in order to make things fair?





Problem of the Week
Problem D and Solution
It Must be Fair!

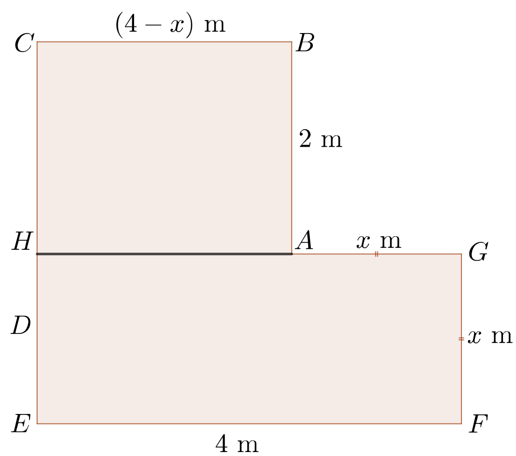
Problem

Two children share an L-shaped bedroom. They are constantly fighting over space. Their parents decide to temporarily partition the room with a curtain so that each child will have exactly the same area, in the hope that the space arguments will end. The layout of the room is represented by $ABCDEFG$ on the diagram. The room has square corners with $EF = 4$ m, $AB = 2$ m, and $AG = GF$. The area of the entire room is $11.2\ \text{m}^2$. The partitioning curtain is to be hung from point A to a point D on CE to divide the room into two parts of equal area. Where is D located on CE to accomplish the equal area split in order to make things fair?

Solution

Let x represent the length of AG. Since $GF = AG$, then $GF = x$.

Extend GA to intersect CE at H. This creates two rectangles $ABCH$ and $GHEF$ with $BC \parallel GH \parallel EF$. Then $BC = EF - AG = 4 - x$.



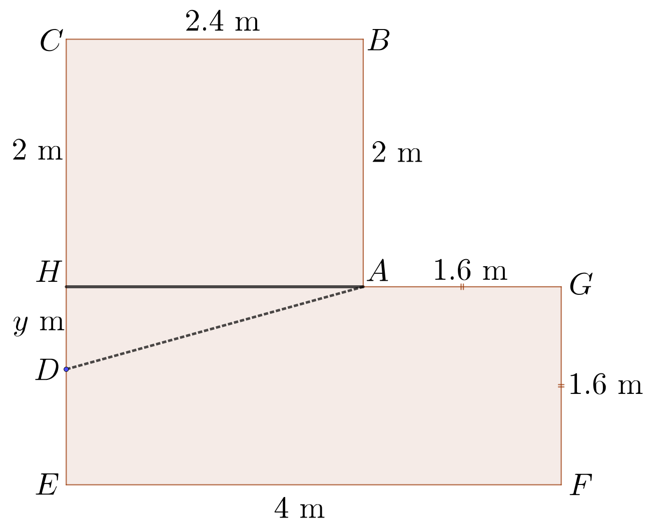
We can now find the value of x using the areas of rectangles.

$$\begin{aligned}
\text{Area } ABCDEFG &= \text{Area } ABCH + \text{Area } GHEF \\
11.2 &= (AB \times BC) + (GF \times EF) \\
11.2 &= 2(4 - x) + x(4) \\
11.2 &= 8 - 2x + 4x \\
3.2 &= 2x \\
1.6 &= x
\end{aligned}$$

Since $x = 1.6$ m, $AG = GF = 1.6$ m and $BC = 4 - x = 2.4$ m. Also, $HC = AB = 2$ m, and $EC = FG + AB = 1.6 + 2 = 3.6$ m.

Let y represent the length of DH.

A diagram with updated information is shown below.



$ABCD$ is a trapezoid with opposite parallel sides $AB = 2$ and $DC = 2 + y$. BC is perpendicular to both AB and DC, and $BC = 2.4$ m. We also know that the area of trapezoid $ABCD$ is half the area of $ABCDEFG$, so the area of trapezoid $ABCD$ is 5.6 m^2. Then,

$$\begin{aligned}
\text{Area of Trapezoid } ABCD &= \frac{BC \times (AB + DC)}{2} \\
5.6 &= \frac{2.4 \times (2 + 2 + y)}{2} \\
5.6 &= 1.2 \times (4 + y) \\
5.6 &= 4.8 + 1.2y \\
0.8 &= 1.2y \\
\frac{0.8}{1.2} &= y \\
\frac{8}{12} &= y \\
\frac{2}{3} &= y
\end{aligned}$$

Since $DC = 2 + y$, $DC = 2 + \frac{2}{3} = \frac{8}{3}$ m.

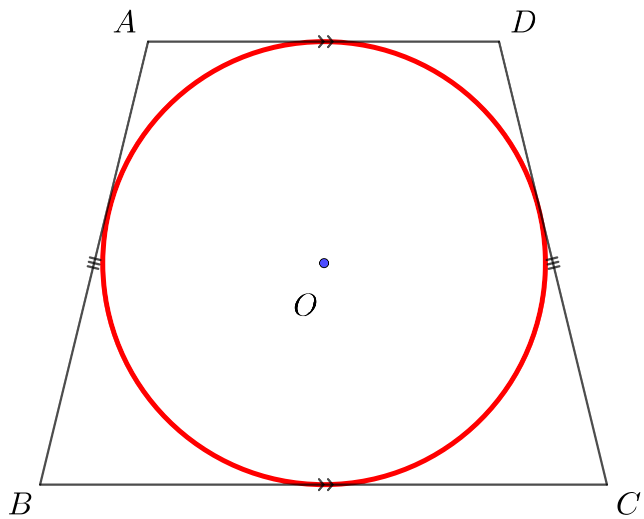
Also, since $ED = EC - DC$, $ED = 3.6 - \frac{8}{3} = \frac{18}{5} - \frac{8}{3} = \frac{54 - 40}{15} = \frac{14}{15}$ m.

The end of the partition located at D should be positioned $\frac{14}{15}$ m from E and $\frac{8}{3}$ m from C.

Problem of the Week
Problem D
Build Around

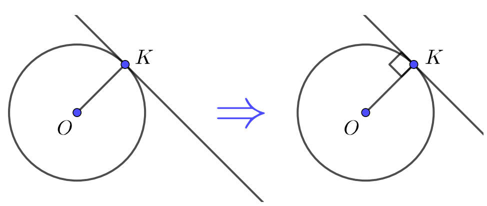
Quadrilateral $ABCD$ is built around a circle with centre O and radius 15 cm so that each side of $ABCD$ is tangent to the circle, sides AD and BC are parallel, and side lengths AB and DC are equal. $ABCD$ is called an isosceles trapezoid.

If the area of $ABCD$ is $1000\,\text{cm}^2$, determine the lengths of the two equal sides, AB and DC.

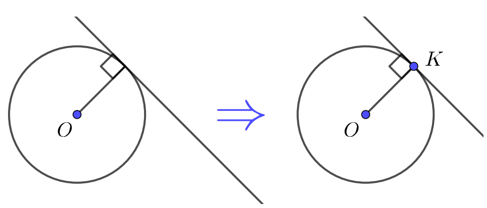


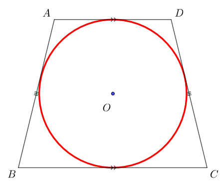
For purposes of this problem, accept the following two results.

- A line segment drawn from the centre of a circle, O, to a point of tangency, K, is perpendicular to the tangent.



- A line segment drawn from the centre of a circle, O, perpendicular to a tangent intersects the tangent at the point of tangency, K.





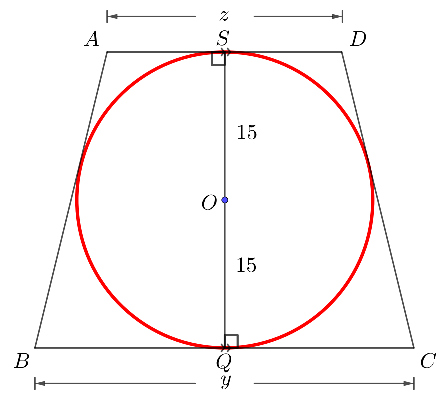
Problem of the Week
Problem D and Solution
Build Around

Problem

Quadrilateral $ABCD$ is built around a circle with centre O and radius 15 cm so that each side of $ABCD$ is tangent to the circle, sides AD and BC are parallel, and side lengths AB and DC are equal. $ABCD$ is called an isosceles trapezoid. If the area of $ABCD$ is $1000\,\text{cm}^2$, determine the lengths of the two equal sides, AB and DC.

Solution

Solution 1



Let z represent the length of side AD and y represent the length of side BC.

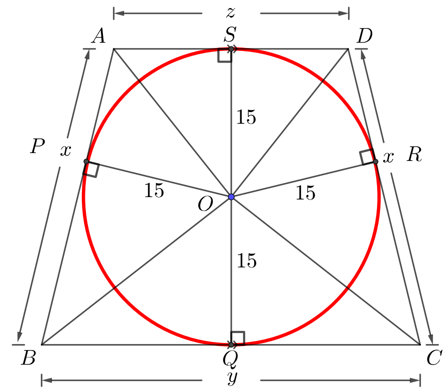
Draw a line segment through O perpendicular to both AD and BC.

Using the second result given after the problem statement, we know that this perpendicular meets AD at the point of tangency S and BC at the point of tangency Q.

Both OS and OQ are radii of the circle. Since $SQ = OS + OQ$, it follows that SQ is a diameter of the circle and has length 30 cm. Since SQ is perpendicular to the two parallel sides of the trapezoid, we can use SQ as the height of the trapezoid.

Using the formula for the area of a trapezoid, we obtain

$$\begin{aligned}
\text{Area} &= \frac{SQ \times (AD + BC)}{2} \\
1000 &= \frac{30 \times (z + y)}{2} \\
1000 &= 15(z + y) \\
\frac{1000}{15} &= z + y \\
\frac{200}{3} &= z + y \quad (1)
\end{aligned}$$



Let the length of the two equal sides, AB and CD, be x.

Join the centre O to each of the vertices of $ABCD$, creating four triangles, $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle DOA$.

Connect the centre O to each of the points of tangency, P, Q, R, and S, on AB, BC, CD, and DA, respectively.

Each of these segments is a radius so $OP = OQ = OR = OS = 15$. From the first result given after the problem statement, we know that each of these segments is perpendicular to the tangent at the point of tangency so each of these radii can be a height of their respective triangles.

We can now find the area of the trapezoid a second way by summing the areas of the four triangles:

$$\text{Area } \triangle AOB = OP \times AB \div 2 = \frac{15x}{2}, \quad \text{Area } \triangle BOC = OQ \times BC \div 2 = \frac{15y}{2}$$

$$\text{Area } \triangle COD = OR \times CD \div 2 = \frac{15x}{2} \quad \text{Area } \triangle DOA = OS \times AD \div 2 = \frac{15z}{2}$$

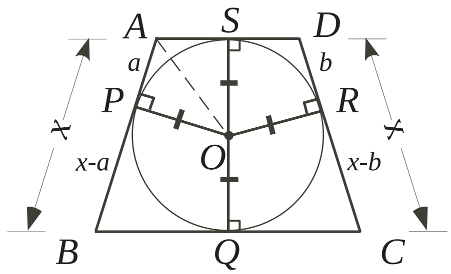
$$\begin{aligned}
\text{Area } \triangle AOB + \text{Area } \triangle BOC + \text{Area } \triangle COD + \text{Area } \triangle DOA &= 1000 \\
\frac{15x}{2} + \frac{15y}{2} + \frac{15x}{2} + \frac{15z}{2} &= 1000 \\
15x + \frac{15y}{2} + \frac{15z}{2} &= 1000 \\
15x + \frac{15y + 15z}{2} &= 1000 \\
15x + \frac{15(y+z)}{2} &= 1000 \\
\text{But } y + z = \frac{200}{3} \text{ from (1) above so} \qquad 15x + \frac{15(\frac{200}{3})}{2} &= 1000 \\
15x + 500 &= 1000 \\
15x &= 500 \\
x &= \frac{100}{3}
\end{aligned}$$

Therefore, the lengths of AB and DC are each $33\frac{1}{3}$ cm.

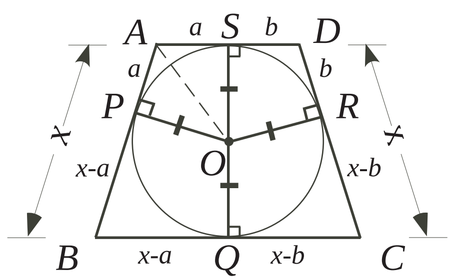
Solution 2

Let P, Q, R, and S be the points of tangency on AB, BC, CD, and DA, respectively. Draw OP, OQ, OR, and OS. Then $OP = OQ = OR = OS = 15$ since they are each radii of the circle. Also, since a line drawn from the centre of the circle to a point of tangency is perpendicular to the tangent, $\angle OPA = \angle OQB = \angle ORD = \angle OSA = 90°$.

Let $AB = x$, $AP = a$ and $DR = b$. Therefore, $DC = x$, $PB = x - a$ and $RC = x - b$. The following diagram shows all of the given and found information.



Join A to O forming two right triangles, $\triangle APO$ and $\triangle ASO$. Using the Pythagorean Theorem, $AP^2 = AO^2 - OP^2$ and $AS^2 = AO^2 - OS^2$. But $OP = OS$ since they are both radii. So the two expressions are equal and $AS = AP = a$ follows.
Using exactly the same reasoning that was used to show $AP = AS = a$, we can show $DR = DS = b$, $BP = BQ = x - a$ and $CR = CQ = x - b$. This new information has been added to the diagram below.



We can now use the area of a trapezoid formula. As was shown in Solution 1, $SQ = SO + OQ$ is a height of the trapezoid. Therefore,

$$\begin{aligned}
\text{Area Trapezoid } ABCD &= SQ \times (AD + BC) \div 2 \\
1000 &= (SO + OQ) \times ((AS + SD) + (BQ + QC)) \div 2 \\
1000 &= (15 + 15) \times ((a + b) + (x - a + x - b)) \div 2 \\
1000 &= (30) \times (2x) \div 2 \\
1000 &= 30x \\
\frac{100}{3} &= x
\end{aligned}$$

Therefore, the lengths of AB and DC are each $33\frac{1}{3}$ cm.

At the end of the statement of the problem, two facts were given. As an extension to this problem, the solver may wish to try to prove these two facts.

Problem of the Week

Problem D

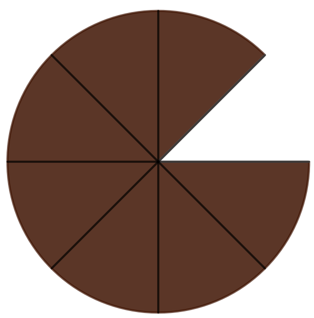
More Cake Please

For Amanda's birthday, Rhett made an amazing, cylindrical, chocolate cream cheesecake. The radius and height of the cake were the same. Rhett cut the cake into 8 congruent slices and ate the first slice for quality control purposes.

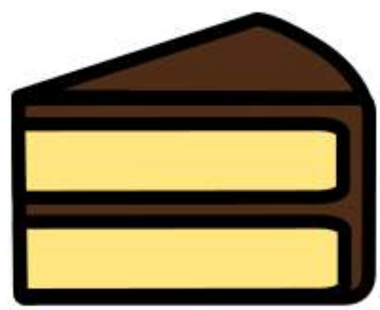
Rhett then posed the following two questions to Amanda:

a) After removing my slice, is there more or less total surface area (top, bottom and all exposed sides) in the remaining cake?

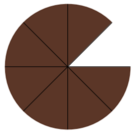
b) By what percentage, to 1 decimal place, has the remaining surface area increased or decreased?



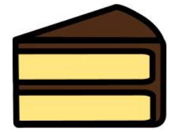
Top View with Rhett's Slice Removed



Rhett's Slice



Problem of the Week
Problem D and Solution
More Cake Please

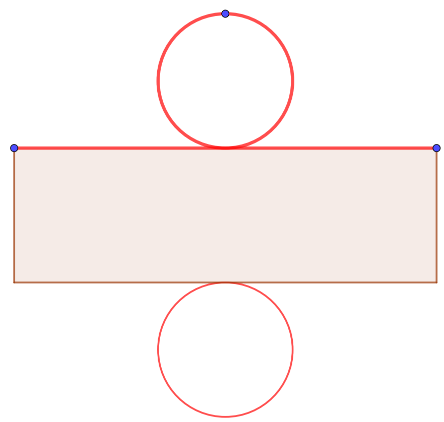


Problem

For Amanda's birthday, Rhett made an amazing, cylindrical, chocolate cream cheesecake. The radius and height of the cake were the same. Rhett cut the cake into 8 congruent slices and ate the first slice for quality control purposes. After removing Rhett's slice, is there more or less total surface area (top, bottom and all exposed sides) in the remaining cake, and by what percentage, to 1 decimal place, has the remaining surface area increased or decreased?

Solution

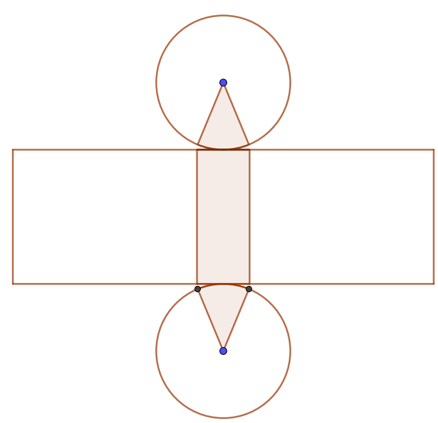
A net illustrating the 3 parts that make up the total surface area is shown to the right.



The total surface area includes the areas of two circles with radius r and a rectangle with length equal to the circumference of the circle and width equal to the height of the cake, $h = r$.

$$\begin{aligned}\text{Total Surface Area} &= 2\left(\pi r^2\right) + (2\pi r)(r)\\ &= 2\pi r^2 + 2\pi r^2\\ &= 4\pi r^2\end{aligned}$$

A net illustrating the 3 parts removed from the total surface area is shown to the right.

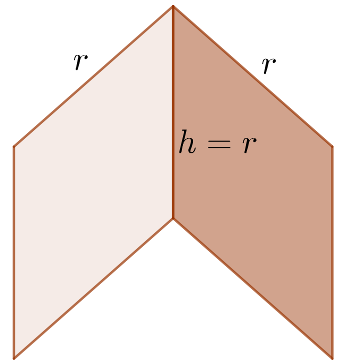


The surface area removed includes $\frac{1}{8}$ of the area of each of two circles with radius r and a rectangle $\frac{1}{8}$ of the area of the original rectangle area. Therefore, the surface area removed is $\frac{1}{8}$ of the total surface area.

$$\begin{aligned}\text{Surface Area Removed} &= \frac{1}{8}\left(4\pi r^2\right)\\ &= \frac{1}{2}\pi r^2\end{aligned}$$

But there are two areas that are added as a result of removing the slice.

A diagram illustrating the 2 parts added to the total surface area is shown to the right.



The surface area added includes 2 rectangles, each with length r and width $h = r$.

$$\begin{aligned} \text{Surface Area Added} &= 2(r)(r) \\ &= 2r^2 \end{aligned}$$

We can now calculate the new total surface area.

$$\begin{aligned} \text{Surface Area} &= \text{Original Surface Area} - \text{Surface Area Removed} + \text{Surface Area Added} \\ &= 4\pi r^2 - \frac{1}{2}\left(\pi r^2\right) + 2r^2 \\ &= 4\pi r^2 + r^2\left(-\frac{1}{2}\pi + 2\right) \end{aligned}$$

Now $\frac{1}{2}\pi < 2$, so $-\frac{1}{2}\pi + 2 > 0$, and the surface area actually increases as a result of removing the slice.

To calculate the percentage that the area has increased, divide the increase by the original area. The increase is $r^2\left(-\frac{1}{2}\pi + 2\right)$.

$$\begin{aligned} \text{Percentage Increase in Area} &= \frac{r^2\left(-\frac{1}{2}\pi + 2\right)}{4\pi r^2} \times 100\% \\ &= \frac{\left(-\frac{1}{2}\pi + 2\right)}{4\pi} \times 100\% \\ &= \left(-\frac{1}{8} + \frac{1}{2\pi}\right) \times 100\% \\ &= \left(\frac{-\pi + 4}{8\pi}\right) \times 100\% \\ &\approx 3.4\% \end{aligned}$$

The surface area of the cake increases by approximately 3.4% after the slice has been removed.

Problem of the Week

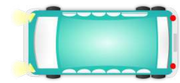
Problem D

Parting Ways

At 7:00 a.m., Sahil drives north at 48 km/h. At the same time from the same intersection, Brenda drives west at 64 km/h.

At what time will they be 260 km apart?

Problem of the Week
Problem D and Solution
Parting Ways



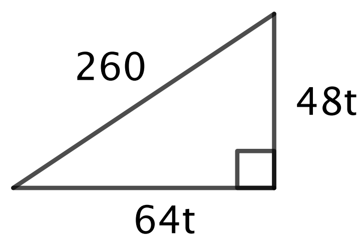
Problem

At 7:00 a.m., Sahil drives north at 48 km/h. At the same time from the same intersection, Brenda drives west at 64 km/h. At what time will they be 260 km apart?

Solution

Let t be the length of time, in hours, that Sahil and Brenda travel until they are 260 km apart. Since Sahil is travelling at 48 km/h, he will travel $48t$ km in t hours. Since Brenda is travelling at 64 km/h, she will travel $64t$ km in t hours.

Since Sahil is travelling north and Brenda is travelling west, they are travelling at right angles to each other. We can represent the distances on the following right triangle.



Using the Pythagorean Theorem

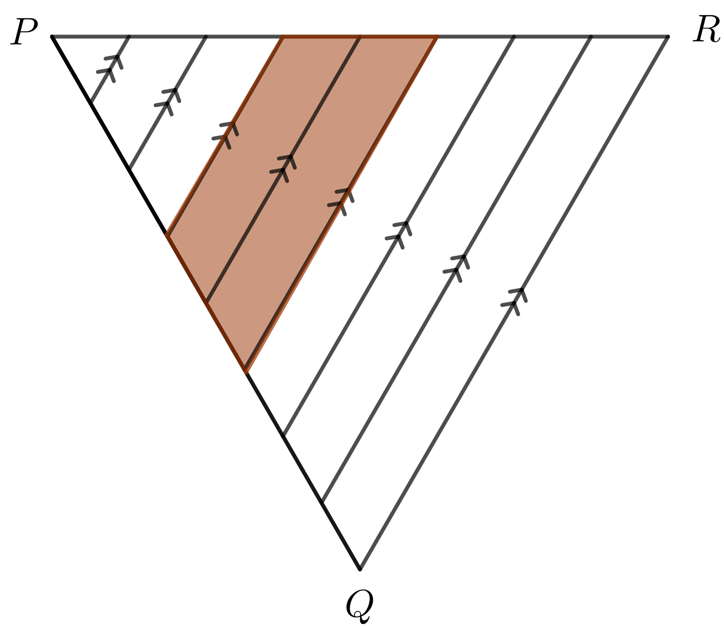
$$\begin{aligned}(48t)^2 + (64t)^2 &= 260^2\\ 2304t^2 + 4096t^2 &= 67600\\ 6400t^2 &= 67600\\ 16t^2 &= 169\\ t^2 &= \frac{169}{16}\end{aligned}$$

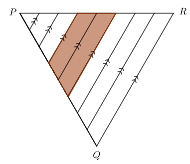
Since $t > 0, t = \frac{13}{4} = 3.25$, which is equivalent to 3 hours and 15 minutes. Note, 3 h 15 min after 7:00 a.m. is 10:15 a.m. Also, $48t = 48 \times \frac{13}{4} = 156$ and $64t = 64 \times \frac{13}{4} = 208$.

Therefore at 10:15 a.m. Sahil and Brenda are 260 km apart. Sahil has travelled 156 km and Brenda has travelled 208 km.

Problem of the Week
Problem D
Shady Area

$\triangle PQR$ is an equilateral triangle with sides of length 32 cm. Two sides of the triangle, PR and PQ, are each divided into 8 segments of equal length. Each point of division on PR is connected to its corresponding point of division on PQ, creating 7 line segments, as shown. Each of the new line segments is parallel to QR, the third side of the triangle. What is the area of the shaded trapezoid?



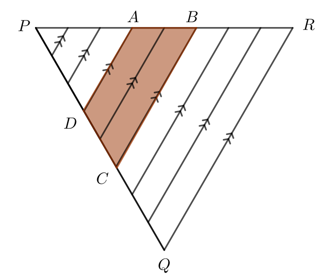


Problem of the Week
Problem D and Solution
Shady Area

Problem

$\triangle PQR$ is an equilateral triangle with sides of length 32 cm. Two sides of the triangle, PR and PQ, are each divided into 8 segments of equal length. Each point of division on PR is connected to its corresponding point of division on PQ, creating 7 line segments, as shown. Each of the new line segments is parallel to QR, the third side of the triangle. What is the area of the shaded trapezoid?

Solution
Solution 1:



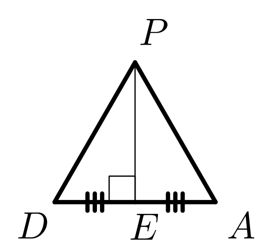
Label the vertices of the trapezoid A, B, C and D, as shown on the diagram. In this solution we will subtract the area of $\triangle PDA$ from the area of $\triangle PCB$ to find the area of trapezoid $ABCD$.

PR and PQ are divided into 8 equal segments, each of length $32 \div 8 = 4$ cm. PD and PA are each made up of 3 of the equal segments, and PC and PB are each made up of 5 of the equal segments. It follows that $PD = PA = 12$ cm and $PC = PB = 20$ cm.

First we will show that $\triangle PDA$ and $\triangle PCB$ are equilateral triangles.
Since $\triangle PQR$ is equilateral, $\angle PRQ = \angle PQR = \angle QPR = 60^\circ$. Since $\angle DPA$, $\angle CPB$ and $\angle QPR$ are the same angle, $\angle DPA = \angle CPD = \angle QPR = 60^\circ$.

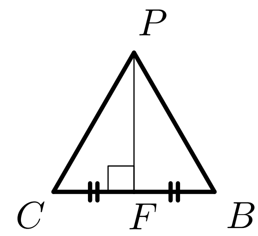
Since $DA \parallel CB \parallel QR$, $\angle PDA = \angle PCB = \angle PQR = 60^\circ$ and $\angle PAD = \angle PBC = \angle PRQ = 60^\circ$.

Since each angle in $\triangle PDA$ and $\triangle PCB$ is 60°, both triangles are equilateral. $\triangle PDA$ has side length 12 cm and $\triangle PBC$ has side length 20 cm.



In $\triangle PDA$, drop a perpendicular from P to E on DA. Since PDA is an equilateral triangle, E is the midpoint of DA and it follows that $DE = \frac{1}{2}DA = 6$ cm. Using the Pythagorean Theorem, $PE^2 = PD^2 - DE^2 = 12^2 - 6^2 = 108$ and $DE = \sqrt{108} = \sqrt{36 \times 3} = 6\sqrt{3}$ cm (since $DE \geq 0$).

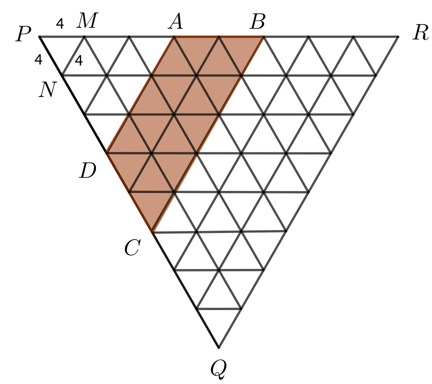
The area of $\triangle PDA = \frac{(PE)(DA)}{2} = \frac{(6\sqrt{3})(12)}{2} = 36\sqrt{3}$ cm^2.



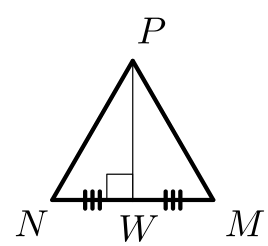
In $\triangle PCB$, drop a perpendicular from P to F on CB. Since PCB is an equilateral triangle, F is the midpoint of CB and it follows that $CF = \frac{1}{2}CB = 10$ cm. Using the Pythagorean Theorem, $PF^2 = PC^2 - CF^2 = 20^2 - 10^2 = 300$ and $PF = \sqrt{300} = \sqrt{100 \times 3} = 10\sqrt{3}$ cm (since $PF \geq 0$).
The area of $\triangle PCB = \frac{(PF)(CB)}{2} = \frac{(10\sqrt{3})(20)}{2} = 100\sqrt{3}$ cm^2.
The area of trapezoid $ABCD$ = area $\triangle PCB$ – area $\triangle PDA = 100\sqrt{3} - 36\sqrt{3} = 64\sqrt{3}$ cm^2.

(Note that we could also have found the lengths of PE and PF by recognizing that $\triangle PDE$ and $\triangle PCF$ are 30°- 60°- 90° triangles with sides in the ratio $1 : \sqrt{3} : 2$.)

Solution 2:



As seen in the diagram to the right, we can tile $\triangle PQR$ with the top left equilateral triangle of side length 4 cm. Three equilateral triangles fit in the first trapezoid from the left, 5 equilateral triangles fit in the second trapezoid from the left, 7 equilateral triangles fit in the third trapezoid from the left, 9 equilateral triangles fit in the fourth trapezoid from the left, and so on. The shaded region contains 16 equilateral triangles, each of which has side length 4 cm. To find the area of the shaded region, we will find the area of an equilateral triangle with side length 4 cm, and multiply the result by 16.



Let the small equilateral triangle be $\triangle PNM$. In $\triangle PNM$, drop a perpendicular from P to W on NM. Since PNM is an equilateral triangle, W is the midpoint of NM and it follows that $NW = \frac{1}{2}NM = 2$ cm.
Using the Pythagorean Theorem, $PW^2 = PN^2 - NW^2 = 4^2 - 2^2 = 12$ and $PW = \sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$ cm (since $PW \geq 0$).
The area of $\triangle PNM = \frac{(PW)(NM)}{2} = \frac{(2\sqrt{3})(4)}{2} = 4\sqrt{3}$ cm^2.

Therefore, the area of the shaded region is $16 \times 4\sqrt{3} = 64\sqrt{3}$ cm^2.

(Note that we could also have found the length of PW by recognizing that $\triangle PNW$ is a 30°- 60°- 90° triangle with sides in the ratio $1 : \sqrt{3} : 2$.)

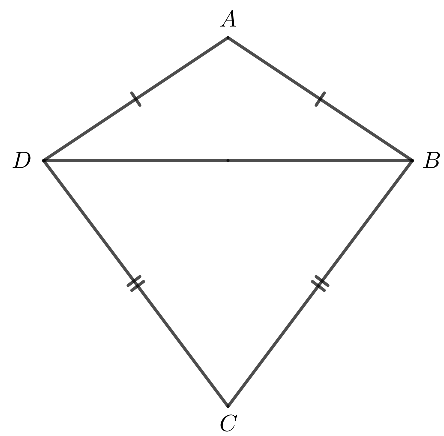
Problem of the Week

Problem D

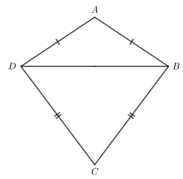
Go Fly a Kite

Amanda wants to fly a kite. The kite is composed of two isosceles triangles, $\triangle ABD$ and $\triangle BCD$. The height of $\triangle BCD$ is 2 times the height of $\triangle ABD$, and the width of the kite, BD, is 1.5 times the height of the larger triangle.

If the area of the kite is 1800 cm^2, what is the perimeter of the kite?



Did you know that in an isosceles triangle the altitude to the unequal side of the triangle bisects that unequal side?



Problem of the Week

Problem D and Solution

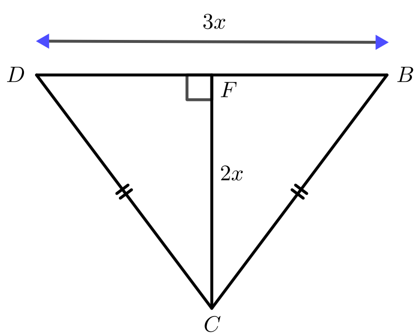
Go Fly a Kite

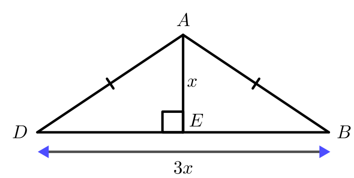
Problem

Amanda wants to fly a kite. The kite is composed of two isosceles triangles, $\triangle ABD$ and $\triangle BCD$. The height of $\triangle BCD$ is 2 times the height of $\triangle ABD$, and the width of the kite, BD, is 1.5 times the height of the larger triangle. If the area of the kite is 1800 cm^2, what is the perimeter of the kite?

Solution

Let the height of $\triangle ABD$ be $AE = x$. Therefore, the height of $\triangle BCD$ is $CF = 2x$. Also, the width of the kite is $BD = 3x$. Therefore, the base of each triangle is $3x$.





The area of $\triangle BCD = \dfrac{(3x)(2x)}{2} = 3x^2$ and the area of $\triangle ABD = \dfrac{(3x)(x)}{2} = \dfrac{3x^2}{2}$.
Also,

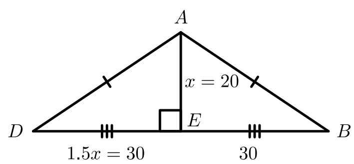
$$\begin{aligned}\text{area of kite } ABCD &= \text{area of } \triangle BCD + \text{area of } \triangle ABD \\ &= 3x^2 + \frac{3x^2}{2} \\ &= \frac{9x^2}{2}\end{aligned}$$

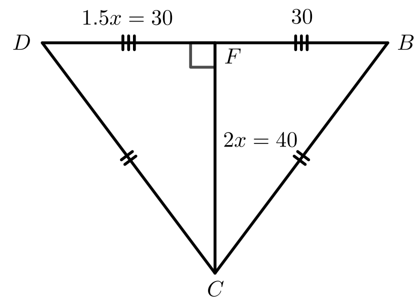
Therefore,

$$\begin{aligned}\frac{9x^2}{2} &= 1800 \\ 9x^2 &= 3600 \\ x^2 &= 400 \\ x &= 20, \quad \text{since } x > 0\end{aligned}$$

Now, to find the perimeter of the kite, we need to find the lengths of the sides of the kite.

Since $\triangle ABD$ is isosceles, E will bisect BD and therefore
$DE = BE = 1.5x = 30$. This is shown below in the diagram to the left.
Similarily, $\triangle BCD$ is isosceles, F will bisect BD, and therefore
$DF = BF = 1.5x = 30$. This is shown below in the diagram to the right.





Using the Pythagorean Theorem in $\triangle AED$,

$$\begin{aligned} AD^2 &= 20^2 + 30^2 \\ &= 400 + 900 \\ &= 1300 \\ AD &= \sqrt{1300}, \text{ since } AD > 0. \end{aligned}$$

Also $AB = AD = \sqrt{1300}$ cm.

Similiarily in $\triangle DFC$,

$$\begin{aligned} DC^2 &= 30^2 + 40^2 \\ &= 2500 \\ DC &= 50, \text{ since } DC > 0. \end{aligned}$$

Also, $BC = DC = 50$ cm.

Now,

$$\begin{aligned} \text{the perimeter of the kite} &= \sqrt{1300} + \sqrt{1300} + 50 + 50 \\ &= 2\sqrt{1300} + 100 \\ &\approx 172.1 \end{aligned}$$

Therefore, the exact perimeter is $2\sqrt{1300} + 100$ cm or approximately 172.1 cm.

Note:

The expression $\sqrt{1300}$ can be simplified as follows:

$$\sqrt{1300} = \sqrt{100 \times 13} = \sqrt{100} \times \sqrt{13} = 10\sqrt{13}.$$

Therefore, the exact perimeter is

$$2\sqrt{1300} + 100 = 2(10\sqrt{13}) + 100 = 20\sqrt{13} + 100 \text{ cm}.$$

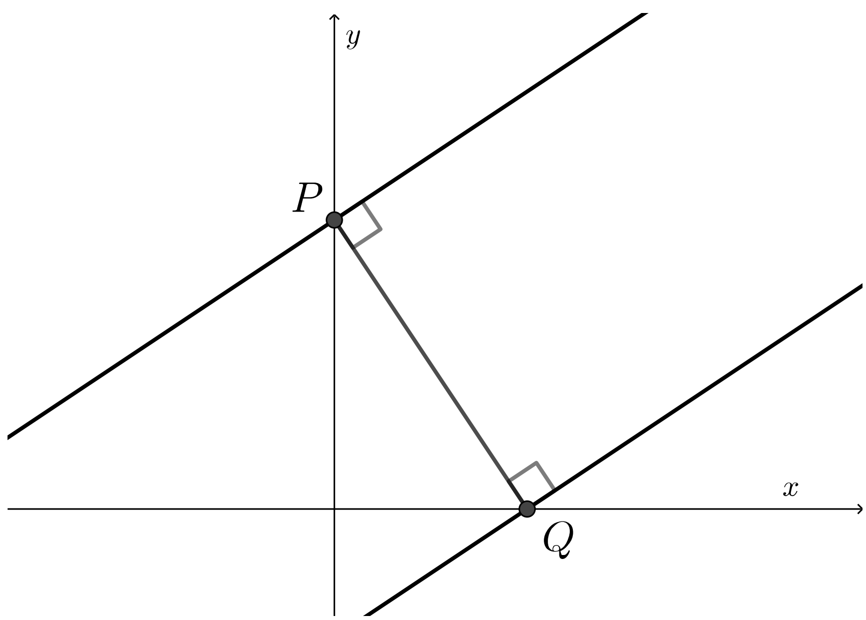
Problem of the Week

Problem D

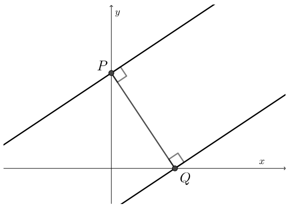
Crossing Points in General

Two distinct lines are drawn such that the first line passes through point P on the y-axis and the second line passes through point Q on the x-axis. Line segment PQ is perpendicular to both lines.

If the line through P has equation $y = mx + k$, then determine the y-intercept of the line through Q in terms of m and k.



Suggestion: If you are finding the general problem difficult to start, consider first solving a problem with a specific example for the line through P, like $y = 4x + 3$, and then attempt the more general problem.



Problem of the Week
Problem D and Solution
Crossing Points in General

Problem

Two distinct lines are drawn such that the first line passes through point P on the y-axis and the second line passes through point Q on the x-axis. Line segment PQ is perpendicular to both lines. If the line through P has equation $y = mx + k$, then determine the y-intercept of line through Q in terms of m and k.

Solution

For ease of reference, we will call the first line l_1 and the second line l_2.

Let b represent the y-intercept of l_2.

Since l_1 has equation $y = mx + k$, we know that the slope of l_1 is m and the y-intercept is k. Therefore, P is the point $(0, k)$.

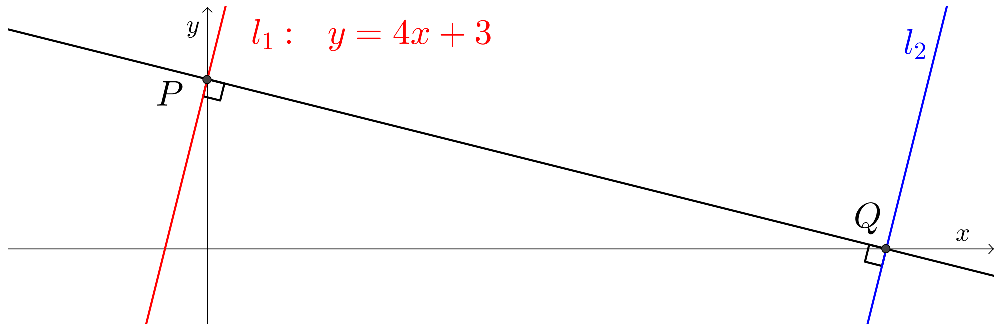
Since PQ is perpendicular to both lines, it follows that l_1 is parallel to l_2. Also, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, $\text{slope}(PQ) = -\frac{1}{m}$. Since k is the y-intercept of the perpendicular segment PQ and the slope of PQ is $-\frac{1}{m}$, the equation of the line through PQ is $y = -\frac{1}{m}x + k$.

To find the x-intercept of $y = -\frac{1}{m}x + k$, set $y = 0$ and solve for x. If $y = 0$, then $0 = -\frac{1}{m}x + k$ and $\frac{1}{m}x = k$. The result $x = mk$ follows. Therefore, the x-intercept of $y = -\frac{1}{m}x + k$ is mk and the coordinates of Q are $(mk, 0)$.

We can now find the y-intercept of l_2 since we know $Q(mk, 0)$ is on l_2 and the slope of l_2 is m. Substituting into the slope-intercept form of the line, $y = mx + b$, we obtain $0 = (m)(mk) + b$ which simplifies to $b = -m^2k$.

Therefore, the y-intercept of l_2, the line through Q, is $-m^2k$.

For the student who solved the problem using $y = 4x + 3$ as the equation of l_1, you should have obtained the answer -48 for the y-intercept of l_2, the line through Q. A full solution to this problem is provided on the next page.



Let l_1 represent the line $y = 4x + 3$. Let l_2 represent the second line, the line through Q.

From the equation of l_1 we know that the slope is 4 and the y-intercept is 3. Therefore P is the point (0,3).

Since $PQ \perp l_1$ and $PQ \perp l_2$, it follows that $l_1 \parallel l_2$. Also, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, slope$(PQ) = -\frac{1}{4}$. Since 3 is the y-intercept of PQ and the slope of PQ is $-\frac{1}{4}$, the equation of the line through PQ is $y = -\frac{1}{4}x + 3$.

The x-intercepts of the line through perpendicular PQ and the line l_2 are the same since both lines intersect at Q on the x-axis. To find this x-intercept, set $y = 0$ in $y = -\frac{1}{4}x + 3$. Then $0 = -\frac{1}{4}x + 3$ and $\frac{1}{4}x = 3$. The result, $x = 12$, follows. The x-intercept of the line through perpendicular PQ and the line l_2 is 12 and point Q is $(12, 0)$.

We can now find equation of l_2 since $Q(12, 0)$ is on l_2 and the slope of l_2 is 4. Substituting $x = 12$, $y = 0$ and $m = 4$ into $y = mx + b$, we obtain $0 = (4)(12) + b$ which simplifies to $b = -48$. The equation of l_2 is $y = 4x - 48$ and the y-intercept is -48.

This is the same result we obtained from the general solution on the previous page.

Algebra (A)

TAKE ME TO THE COVER

Problem of the Week
Problem D
An Average Report

A report card has six marks. The average of the first and second marks is 72%. The average of the second and third marks is 75%. The average of the third and fourth marks is 77%. The average of the fourth and fifth marks is 78%. The average of the fifth and sixth marks is 79%.

(a) Determine the overall average.

(b) Determine the average of the first and sixth marks.

Problem of the Week
Problem D and Solution
An Average Report

Problem

A report card has six marks. The average of the first and second marks is 72%. The average of the second and third marks is 75%. The average of the third and fourth marks is 77%. The average of the fourth and fifth marks is 78%. The average of the fifth and sixth marks is 79%. a) Determine the overall average. b) Determine the average of the first and sixth marks.

Solution

Let a, b, c, d, e, f represent the six report card marks.

The average of the first and second marks is 72, so $\frac{a+b}{2} = 72$. Multiplying by 2, $a + b = 144$. (1)

The average of the second and third marks is 75, so $\frac{b+c}{2} = 75$, leading to $b + c = 150$. (2)

The average of the third and fourth marks is 77, so $\frac{c+d}{2} = 77$, leading to $c + d = 154$. (3)

The average of the fourth and fifth marks is 78, so $\frac{d+e}{2} = 78$, leading to $d + e = 156$. (4)

The average of the fifth and sixth marks is 79, so $\frac{e+f}{2} = 79$, leading to $e + f = 158$. (5)

(a) To find the overall average we must find the sum $a + b + c + d + e + f$ and divide by 6. If we add equations (1), (3) and (5) we obtain the required sum.

$$\begin{aligned}(a+b)+(c+d)+(e+f) &= 144+154+158\\ a+b+c+d+e+f &= 456\\ \frac{a+b+c+d+e+f}{6} &= 76\end{aligned}$$

$\therefore$ The overall average is 76%.

(b) To find the average of the first and sixth marks, we must find the sum $a + f$ and divide by 2. We will add equations (1), (2), (3), (4) and (5).

$$\begin{aligned}(a+b)+(b+c)+(c+d)+(d+e)+(e+f) &= 144+150+154+156+158\\ a+2b+2c+2d+2e+f &= 762\\ a+2(b+c)+2(d+e)+f &= 762\\ a+2(150)+2(156)+f &= 762, \text{substituting from (2) and (4)}\\ a+300+312+f &= 762\\ a+f &= 150\\ \frac{a+f}{2} &= 75\end{aligned}$$

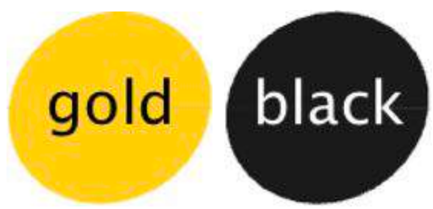
$\therefore$ the average of the first and sixth marks is 75%.

Problem of the Week
Problem D
Go for the Gold

For orientation days at the University of Waterloo, various activities are available for incoming students to participate in. One activity is a simple game in which students reach into a box and randomly select a golf ball. The golf balls are all identical in size and are either black or gold. If a student selects a gold golf ball, then they win a prize. After a golf ball is selected, it is returned to the box.

Initially, the box contained 300 gold golf balls and students had a 1 in 5 chance of selecting a gold golf ball. The organizers want to increase the chances of selecting a gold golf ball to 3 in 10. In order to do this, they add complete packages of golf balls. Each package of golf balls contains 60 golf balls, of which 65% are black.

How many full packages of golf balls must be added to the box?



gold

Problem of the Week
Problem D and Solution
Go for the Gold

black

Problem

For orientation days at the University of Waterloo, various activities are available for incoming students to participate in. One activity is a simple game in which students reach into a box and randomly select a golf ball. The golf balls are all identical in size and are either black or gold. If a student selects a gold golf ball, then they win a prize. After a golf ball is selected, it is returned to the box. Initially, the box contained 300 gold golf balls and students had a 1 in 5 chance of selecting a gold golf ball. The organizers want to increase the chances of selecting a gold golf ball to 3 in 10. In order to do this, they add complete packages of golf balls. Each package of golf balls contains 60 golf balls, of which 65% are black. How many full packages of golf balls must be added to the box?

Solution

There were initially 300 gold golf balls in the box and the students had a 1 in 5 chance of winning. This means that there was is 4 in 5 chance of selecting a black golf ball. Therefore, there were four times as many black golf balls as gold golf balls. That is, there were $4 \times 300 = 1200$ black golf balls and a total of $300 + 1200 = 1500$ golf balls in the box before any new packages were added.

Each new package of golf balls contains 60 golf balls. Since 65% are black, there are $0.65 \times 60 = 39$ black golf balls and $60 - 39 = 21$ gold golf balls in each new package.

Let n represent the number of packages of golf balls added to the box to increase the chances of winning from 1 in 5 to 3 in 10. By adding n packages of golf balls to the box, we are adding $21n$ gold golf balls and $60n$ golf balls to the box. After adding the packages, the box will contain $300 + 21n$ gold golf balls and $1500 + 60n$ golf balls. Since the chances of winning are now 3 in 10,

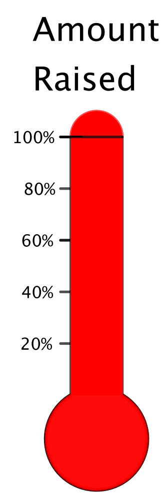
$$\begin{aligned}
\frac{\text{the number of gold golf balls}}{\text{the total number of golf balls}} &= \frac{3}{10}\\
\frac{300 + 21n}{1500 + 60n} &= \frac{3}{10}\\
10(300 + 21n) &= 3(1500 + 60n)\\
3000 + 210n &= 4500 + 180n\\
30n &= 1500\\
n &= 50
\end{aligned}$$

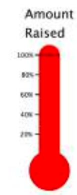
Therefore, 50 new packages of golf balls must be added to the box to raise the chances of winning from 1 in 5 to 3 in 10.

We can check this easily. After adding 50 new packages of golf balls, there would be $300 + 21(50) = 1350$ gold golf balls and a total of $1500 + 60(50) = 4500$ golf balls. Then, the ratio of gold golf balls to the total number of golf balls in the box is $\frac{1350}{4500} = \frac{3}{10}$, as required.

Problem of the Week
Problem D
Goal Reached!

Don Ater and Cole Lector have a team fundraising goal that they are determined to reach. The team goal is 60% more than the amount Don has raised and 80% more than the amount Cole has raised. After Don and Cole combine the amounts they collected, what percentage (correct to one decimal place) above their goal did they collect?





Problem of the Week

Problem D and Solution

Goal Reached!

Problem

Don Ater and Cole Lector have a team fundraising goal that they are determined to reach. The team goal is 60% more than the amount Don has raised and 80% more than the amount Cole has raised. After Don and Cole combine the amounts they collected, what percentage (correct to one decimal place) above their goal did they collect?

Solution

Let g be the fundraising goal in \$, d be Don's amount raised in \$, c be Cole's amount raised in \$.

Since the team goal is 60% more than the amount Don raised, then $g = 1.60d = \frac{8}{5}d$, or $d = \frac{5}{8}g$.

Similarly, since the team goal is 80% more than the amount Cole raised, $g = 1.80c = \frac{9}{5}c$, or $c = \frac{5}{9}g$.

Let t be the total raised by Don and Cole, in \$.

Now,

$$\begin{aligned} t &= d + c \\ &= \frac{5}{8}g + \frac{5}{9}g \\ &= \left(\frac{5}{8} + \frac{5}{9}\right)g \\ &= \left(\frac{45}{72} + \frac{40}{72}\right)g \\ &= \frac{85}{72}g \\ &= \left(1 + \frac{13}{72}\right)g \end{aligned}$$

Therefore, the total raised exceeds the fundraising goal by $\frac{13}{72} \times 100\%$, which is approximately 18.1%.

Problem of the Week
Problem D
Thumbs Up for a Job Well Done

A team of employees completed a large project. Their manager gave them a big thumbs up. Later they were recognized more tangibly with a monetary bonus to share among themselves. The ages of the team members are consecutive integers and no one on the team has the same age. The oldest team member is 45.

The bonus was paid out as follows:

(i) \$1000 to the oldest member of the project team plus $\frac{1}{10}$ of what remains, then

(ii) \$2000 to the second oldest member of the project team plus $\frac{1}{10}$ of what then remains, then

(iii) \$3000 to the third oldest member of the project team plus $\frac{1}{10}$ of what then remains, and so on.

After all of the bonus money had been distributed, each member of the project team had received the same amount. What is the age of the youngest member of the project team?

Problem of the Week
Problem D and Solution
Thumbs Up for a Job Well Done

Problem

A team of employees completed a large project. Their manager gave them a big thumbs up. Later they were recognized more tangibly with a monetary bonus to share among themselves. The ages of the team members are consecutive integers and no one on the team has the same age. The oldest team member is 45.

The bonus was paid out as follows: \$1000 to the oldest member of the project team plus $\frac{1}{10}$ of what remains, then \$2000 to the second oldest member of the project team plus $\frac{1}{10}$ of what then remains, then \$3000 to the third oldest member of the project team plus $\frac{1}{10}$ of what then remains, and so on. After all of the bonus money had been distributed, each member of the project team had received the same amount. What is the age of the youngest member of the project team?

Solution

Solution 1

Let x represent the amount that each team member receives.
Let y represent the total amount of the bonus.
Then $y \div x$ is the number of team members.

The first team member gets \$1000 plus one-tenth of the remainder:

$$\begin{array}{rrcll} & x & = & 1000 + \frac{1}{10}(y - 1000) & \\ \text{Multiply both sides by 10:} & 10x & = & 10\,000 + y - 1000 & \\ \text{Simplify and solve for } y: & y & = & 10x - 9000 & (1) \end{array}$$

The second team member gets \$2000 plus one-tenth of the remainder after the first team member's share and \$2000 is removed:

$$\begin{array}{rrcll} & x & = & 2000 + \frac{1}{10}(y - x - 2000) & \\ \text{Multiply both sides by 10:} & 10x & = & 20\,000 + y - x - 2000 & \\ \text{Simplify and solve for } y: & y & = & 11x - 18\,000 & (2) \\ \text{Since } y = y \text{ in (1) and (2):} & 10x - 9000 & = & 11x - 18000 & \\ & \therefore x & = & 9000 & \\ \text{Substitute for } x \text{ in (1):} & y & = & 10(9000) - 9000 & \\ & & = & 81\,000 & \end{array}$$

Each team member receives \$9000 and the total bonus is \$81 000. The number of team members is $y \div x = 81\ 000 \div 9000 = 9$. The oldest team member is 45 and the ages of the team members are consecutive integers, so the youngest team member is 37.

Solution 2

Let n represent the number of team members.

Team member n (the youngest team member) receives $n \times \$1000$ or $1000n$. (1)

So $1000n$ is $\frac{9}{10}$ of what remains after team member $(n-1)$ is given $(n-1) \times 1000$.

$\therefore \frac{1}{10}$ of what remains is $\frac{1000n}{9}$.

Then team member $(n-1)$ receives $1000(n-1) + \frac{1000n}{9}$.

But team member $(n-1)$ and team member n each receive the same amount.

Therefore,

$$\begin{aligned}
1000(n-1) + \frac{1000n}{9} &= 1000n \\
1000n - 1000 + \frac{1000n}{9} &= 1000n \\
\frac{1000n}{9} &= 1000 \\
1000n &= 9000 \\
n &= 9 \\
\text{Substituting in (1):} \qquad 1000n &= 9\,000
\end{aligned}$$

There are 9 team members and each receives $\$9\,000$. Since, the oldest team member is 45 and the ages of the team members are consecutive integers, the youngest team member is 37.

It is easy to think that the correct age of the youngest team member would be $45 - 9 = 36$. This actually turns out to be 10 people.

The ages of the group members are

$$37 \quad 38 \quad 39 \quad 40 \quad 41 \quad 42 \quad 43 \quad 44 \quad 45$$

There are 9 integers from 37 to 45, inclusive.

Problem of the Week

Problem D

One Step at a Time

Sequences of numbers can be generated by following a variety of steps to get from term to the next term.

Our particular sequence begins with first term 2. To obtain the next term in the sequence from the term immediately before it, multiply the preceding term by 3, then add 2 to your result and finally divide by this new result by 3. Repeat this set of steps with each new term to generate more terms in the sequence.

For example, the second term in our sequence would be calculated as follows:

$$2 \Rightarrow \text{multiply by } 3 \Rightarrow 6 \Rightarrow \text{add } 2 \Rightarrow 8 \Rightarrow \text{divide by } 3 \Rightarrow \frac{8}{3}$$

The value of the second term is $\frac{8}{3}$ and the term number is 2.

If you follow the same steps with term 2, you will obtain term 3 whose value is $\frac{10}{3}$.

Determine the value of the 1000^{th} term.

Term Number	1	2	3	$\cdots\quad\cdots$	1000
Value	2	$\frac{8}{3}$	$\frac{10}{3}$	$\cdots\quad\cdots$	????

Problem of the Week
Problem D and Solution
One Step at a Time

Problem

A particular sequence begins with first term 2. To obtain the next term in the sequence from the term immediately before it, multiply the preceding term by 3, then add 2 to your result and finally divide by this new result by 3. Repeat this set of steps with each new term to generate more terms in the sequence.

Following these steps with first term 2, the value of the second term is $\frac{8}{3}$ and the term number is 2. Following the steps with the second term, you will obtain term 3 whose value is $\frac{10}{3}$. Determine the value of the 1000^{th} term.

Term Number	1	2	3	$\cdots\ \cdots$	1000
Value	2	$\frac{8}{3}$	$\frac{10}{3}$	$\cdots\ \cdots$	????

Solution

Solution 1

In order to find the 1000^{th} term, when using the definition, we would need the 999^{th} term. To find the 999^{th} term, we would need the 998^{th} term. And to find the 998^{th} term, we would need the 997^{th} term; and so on. Using this method is not practical but it would lead to the 1000^{th} term 668.

However, we can sometimes develop a rule that allows us to determine the value of a term that only depends on knowing the term number (its position in the sequence). We will do this in the following solutions.

Solution 2

To start, we will generate a few more terms.

For term 4: $\frac{10}{3} \Rightarrow \boxed{\text{Multiply by 3}} \Rightarrow 10 \Rightarrow \boxed{\text{Add 2}} \Rightarrow 12 \Rightarrow \boxed{\text{Divide by 3}} \Rightarrow \frac{12}{3} = 4$

For term 5: $4 \Rightarrow \boxed{\text{Multiply by 3}} \Rightarrow 12 \Rightarrow \boxed{\text{Add 2}} \Rightarrow 14 \Rightarrow \boxed{\text{Divide by 3}} \Rightarrow \frac{14}{3}$

For term 6: $\frac{14}{3} \Rightarrow \boxed{\text{Multiply by 3}} \Rightarrow 14 \Rightarrow \boxed{\text{Add 2}} \Rightarrow 16 \Rightarrow \boxed{\text{Divide by 3}} \Rightarrow \frac{16}{3}$

For term 7: $\frac{16}{3} \Rightarrow \boxed{\text{Multiply by 3}} \Rightarrow 16 \Rightarrow \boxed{\text{Add 2}} \Rightarrow 18 \Rightarrow \boxed{\text{Divide by 3}} \Rightarrow \frac{18}{3} = 6$

The sequence begins $2, \frac{8}{3}, \frac{10}{3}, 4, \frac{14}{3}, \frac{16}{3}, 6, \cdots$.

Notice that the difference between consecutive terms appears to always be $\frac{2}{3}$. We will present a justification as to why this is true here.

If p is any term in the sequence and q is the term immediately following it, then we get q by multiplying p by 3, adding 2 to the product, and then dividing the result by 3. That is, $q = \frac{3p+2}{3} = \frac{3p}{3} + \frac{2}{3} = p + \frac{2}{3}$. Then $q = p + \frac{2}{3}$ or $q - p = \frac{2}{3}$. That is, the difference between consecutive terms is $\frac{2}{3}$.

Term 1 and term 1000 are 999 terms apart. So, to get from term 1 to term 1000, we would add 999 multiples of $\frac{2}{3}$ to the first term. Therefore, the 1000^{th} term is $2 + 999 \times \frac{2}{3} = 2 + 666 = 668$.

Solution 3

We will start this solution with the sequence of terms generated in solution 2.

The sequence is $2, \frac{8}{3}, \frac{10}{3}, 4, \frac{14}{3}, \frac{16}{3}, 6, \cdots$.

Notice that the difference between consecutive terms is $\frac{2}{3}$. A justification of this was presented in Solution 2.

Also note that terms 1, 4 and 7 are the positive integers 2, 4 and 6. The value of any term in these positions is 2 greater than the value of the term 3 positions before it. In fact, the value of any term in position $(1 + 3n)$ for non-negative integer values of n will be $(2 + 2n)$.

We are asked for term 1000. The n that satisfies $1 + 3n = 1000$ is $n = 333$.

When $n = 333$, $2 + 2n = 2 + 2(333) = 668$. Therefore, the 1000^{th} term is 668.

Solution 4

Using the terms generated in solution 2, we can create the following chart.

Term Number	1	2	3	4	5	6	7
Value	$2 = \frac{6}{3}$	$\frac{8}{3}$	$\frac{10}{3}$	$4 = \frac{12}{3}$	$\frac{14}{3}$	$\frac{16}{3}$	$6 = \frac{18}{3}$

Notice that the first, fourth and seventh terms have been re-written as a fraction with denominator 3.

Looking at each term, observe that the denominators are all 3 and the numerators increase by 2. To get to any term in the sequence, we could add 2 times the term number to the numerator 4 and keep the denominator as 3. That is, if n is the term number, then term n, usually written t_n, would be $\frac{4+2n}{3}$.

For the 1000^{th} term, we substitute $n = 1000$ into the expression for the general term. It follows that the 1000^{th} term is $t_{1000} = \frac{4+2(1000)}{3} = \frac{2004}{3} = 668$.

Solution 5

Let x represent the term number such that x a a positive integer.

Let y represent the value of the term in position x.

x	1	2	3	4	5	6	7
y	2	$\frac{8}{3}$	$\frac{10}{3}$	4	$\frac{14}{3}$	$\frac{16}{3}$	6

As the value of the independent variable x increases by 1, the value of the dependent variable y increases by $\frac{2}{3}$. This data can therefore be modelled with a linear function of the form $y = mx + b$.

Using $(1, 2)$ and $(4, 4)$, we can determine that $m = \dfrac{4-2}{4-1} = \dfrac{2}{3}$.

Substituting $x = 1,\ y = 2,$ and $m = \dfrac{2}{3}$ into

$$\begin{aligned}
& & y &= mx + b \\
&\text{we obtain:} & 2 &= \frac{2}{3}(1) + b \\
&\text{multiplying by 3:} & 6 &= 2 + 3b \\
&\text{solving for } b\text{:} & 4 &= 3b \\
& & \frac{4}{3} &= b
\end{aligned}$$

So, for positive integer values of x, we can use $y = \dfrac{2}{3}x + \dfrac{4}{3}$ to generate terms in the sequence.

For the value of the 1000^{th} term, substitute $x = 1000$ into $y = \dfrac{2}{3}x + \dfrac{4}{3}$.

Then $y = \dfrac{2}{3}(1000) + \dfrac{4}{3} = \dfrac{2000}{3} + \dfrac{4}{3} = \dfrac{2004}{3} = 668$.

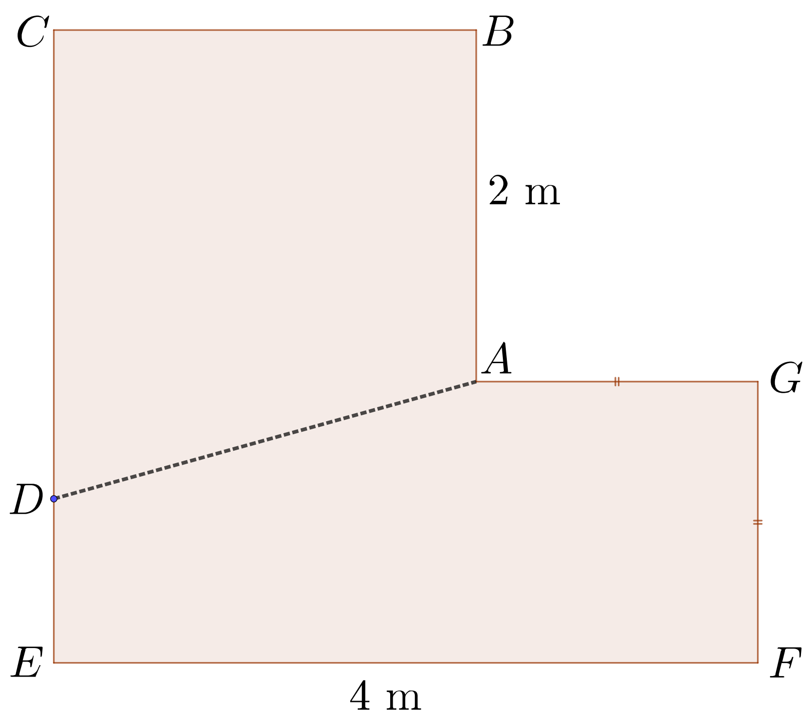
Therefore the 1000^{th} term in the sequence is 668.

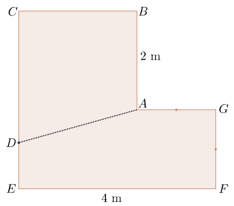
Note, you may see this general term written $t_n = \dfrac{2n+4}{3}$ where n, the term number, is a positive integer and t_n represents the value of the term in that position.

Problem of the Week
Problem D
It Must be Fair!

Two children share an L-shaped bedroom. They are constantly fighting over space. Their parents decide to temporarily partition the room with a curtain so that each child will have exactly the same area, in the hope that the space arguments will end.

The layout of the room is represented by $ABCDEFG$ on the diagram. The room has square corners with $EF = 4$ m, $AB = 2$ m, and $AG = GF$. The area of the entire room is 11.2 m^2. The partitioning curtain is to be hung from point A to a point D on CE to divide the room into two parts of equal area. Where is D located on CE to accomplish the equal area split in order to make things fair?





Problem of the Week
Problem D and Solution
It Must be Fair!

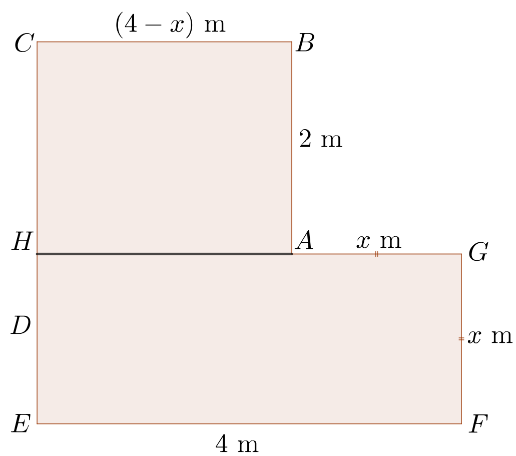
Problem

Two children share an L-shaped bedroom. They are constantly fighting over space. Their parents decide to temporarily partition the room with a curtain so that each child will have exactly the same area, in the hope that the space arguments will end. The layout of the room is represented by $ABCDEFG$ on the diagram. The room has square corners with $EF = 4$ m, $AB = 2$ m, and $AG = GF$. The area of the entire room is $11.2\ \text{m}^2$. The partitioning curtain is to be hung from point A to a point D on CE to divide the room into two parts of equal area. Where is D located on CE to accomplish the equal area split in order to make things fair?

Solution

Let x represent the length of AG. Since $GF = AG$, then $GF = x$.

Extend GA to intersect CE at H. This creates two rectangles $ABCH$ and $GHEF$ with $BC \parallel GH \parallel EF$. Then $BC = EF - AG = 4 - x$.



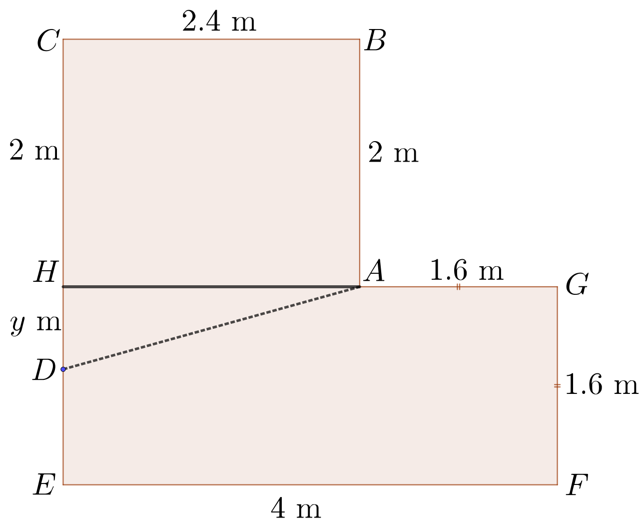
We can now find the value of x using the areas of rectangles.

$$\begin{aligned}
\text{Area } ABCDEFG &= \text{Area } ABCH + \text{Area } GHEF \\
11.2 &= (AB \times BC) + (GF \times EF) \\
11.2 &= 2(4 - x) + x(4) \\
11.2 &= 8 - 2x + 4x \\
3.2 &= 2x \\
1.6 &= x
\end{aligned}$$

Since $x = 1.6$ m, $AG = GF = 1.6$ m and $BC = 4 - x = 2.4$ m. Also, $HC = AB = 2$ m, and $EC = FG + AB = 1.6 + 2 = 3.6$ m.

Let y represent the length of DH.

A diagram with updated information is shown below.



$ABCD$ is a trapezoid with opposite parallel sides $AB = 2$ and $DC = 2 + y$. BC is perpendicular to both AB and DC, and $BC = 2.4$ m. We also know that the area of trapezoid $ABCD$ is half the area of $ABCDEFG$, so the area of trapezoid $ABCD$ is 5.6 m^2. Then,

$$\begin{aligned}
\text{Area of Trapezoid } ABCD &= \frac{BC \times (AB + DC)}{2} \\
5.6 &= \frac{2.4 \times (2 + 2 + y)}{2} \\
5.6 &= 1.2 \times (4 + y) \\
5.6 &= 4.8 + 1.2y \\
0.8 &= 1.2y \\
\frac{0.8}{1.2} &= y \\
\frac{8}{12} &= y \\
\frac{2}{3} &= y
\end{aligned}$$

Since $DC = 2 + y$, $DC = 2 + \frac{2}{3} = \frac{8}{3}$ m.

Also, since $ED = EC - DC$, $ED = 3.6 - \frac{8}{3} = \frac{18}{5} - \frac{8}{3} = \frac{54 - 40}{15} = \frac{14}{15}$ m.

The end of the partition located at D should be positioned $\frac{14}{15}$ m from E and $\frac{8}{3}$ m from C.

Problem of the Week
Problem D
This is the Year

The positive integers can be arranged as follows.

Row 1	1					
Row 2	2	3				
Row 3	4	5	6			
Row 4	7	8	9	10		
Row 5	11	12	13	14	15	
Row 6	16	17	18	19	20	21
$\vdots$						

More rows and columns continue to list the positive integers in order, with each new row containing one more integer than the previous row.

How many integers less than 2020 are in the *column* that contains the number 2020?

Did you know that the sum of the positive integers from 1 to n can be determined using the formula $\frac{n(n+1)}{2}$? That is, $1+2+3+4+\ldots+(n-1)+n=\frac{n(n+1)}{2}$. For example, the sum of the integers $1+2+3+4=\frac{4(5)}{2}=10$. This result can be verified by simply adding the 4 numbers. You can also easily verify that the sum of the first 5 positive integers is $\frac{5(6)}{2}=15$.

This formula may be useful in solving this problem. As an extension, one may wish to prove this formula holds for any positive integer n.

Problem of the Week
Problem D and Solution
This is the Year

Problem
The positive integers can be arranged as follows.

Row 1	1					
Row 2	2	3				
Row 3	4	5	6			
Row 4	7	8	9	10		
Row 5	11	12	13	14	15	
Row 6	16	17	18	19	20	21
⋮						

More rows and columns continue to list the positive integers in order, with each new row containing one more integer than the previous row. How many integers less than 2020 are in the *column* that contains the number 2020?

Solution

In the table given, there is one number in Row 1, there are two numbers in Row 2, three numbers in Row 3, and so on, with n numbers in Row n.
The numbers in the rows list the positive integers in order beginning at 1 in Row 1, with each new row containing one more integer than the previous row. Thus, the last number in each row is equal to the sum of the number of numbers in each row of the table up to that row.

For example, the last number in Row 4 is 10, which is equal to the sum of the number of numbers in rows 1, 2, 3, and 4. But the number of numbers in each row is equal to the row number. So 10 is equal to the sum $1 + 2 + 3 + 4$.
That is, the last number in Row n is equal to the sum

$$1 + 2 + 3 + 4 + \ldots + (n-1) + n = \tfrac{n(n+1)}{2}.$$

To find out which row the integer 2020 occurs in, we could use organized trial and error.
Using trial and error, we find that since $\frac{63(63+1)}{2} = 2016$, then the last number in Row 63 is 2016.
We then find $\frac{64(64+1)}{2} = 2080$, so the last number in Row 64 is 2080.
Since 2020 is between 2016 and 2080, then it must appear somewhere in the 64th row.

Alternatively, we could determine the row the integer 2020 occurs in by using the

quadratic formula to solve the equation $\frac{n(n+1)}{2} = 2020$ for n.

$$\begin{aligned}
\frac{n(n+1)}{2} &= 2020 \\
n(n+1) &= 4040 \\
n^2 + n - 4040 &= 0 \\
n &\approx 63.1 \text{ (using the quadratic formula and } n \geq 0)
\end{aligned}$$

This means that the integer 2020 will be in Row 64.

If we look at the original arrangement given, the first number in Row 6, which is 16, has five numbers in the column above it. The second number in Row 6, which is 17, has four numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.

Also, note that the first number in Row 5, which is 11, has four numbers in the column above it. The second number in Row 5, which is 12, has three numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.

The pattern is that the first number in Row n has $n - 1$ numbers in the column above it. The second number in Row n has $n - 2$ numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.

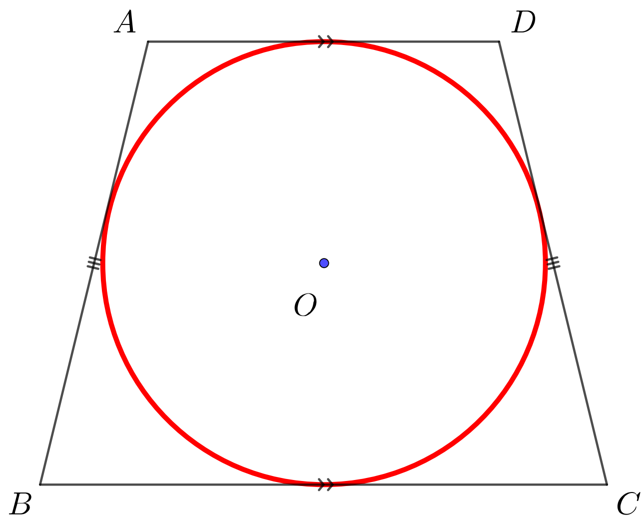
Therefore, the first number of Row 64, which is 2017, will have 63 numbers in the column above it. The second number of Row 64, which is 2018, will have 62 numbers in the column above it. The third number of Row 64, which is 2019, will have 61 numbers in the column above it. The fourth number of Row 64, which is 2020, will have 60 numbers in the column above it.

Therefore, there are 60 integers less than 2020 in the column that contains the number 2020.

Problem of the Week
Problem D
Build Around

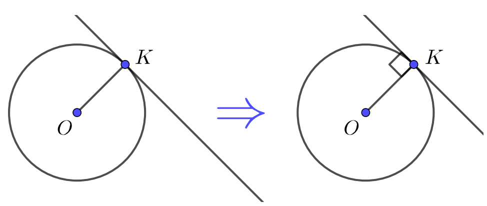
Quadrilateral $ABCD$ is built around a circle with centre O and radius 15 cm so that each side of $ABCD$ is tangent to the circle, sides AD and BC are parallel, and side lengths AB and DC are equal. $ABCD$ is called an isosceles trapezoid.

If the area of $ABCD$ is $1000\,\text{cm}^2$, determine the lengths of the two equal sides, AB and DC.

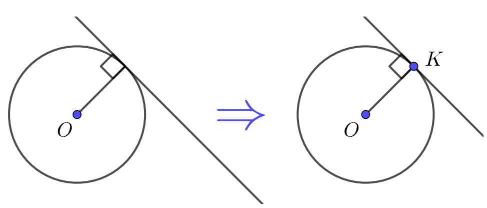


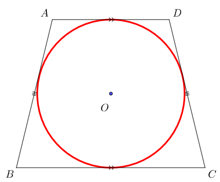
For purposes of this problem, accept the following two results.

- A line segment drawn from the centre of a circle, O, to a point of tangency, K, is perpendicular to the tangent.



- A line segment drawn from the centre of a circle, O, perpendicular to a tangent intersects the tangent at the point of tangency, K.





Problem of the Week
Problem D and Solution
Build Around

Problem

Quadrilateral $ABCD$ is built around a circle with centre O and radius 15 cm so that each side of $ABCD$ is tangent to the circle, sides AD and BC are parallel, and side lengths AB and DC are equal. $ABCD$ is called an isosceles trapezoid. If the area of $ABCD$ is $1000\,\text{cm}^2$, determine the lengths of the two equal sides, AB and DC.

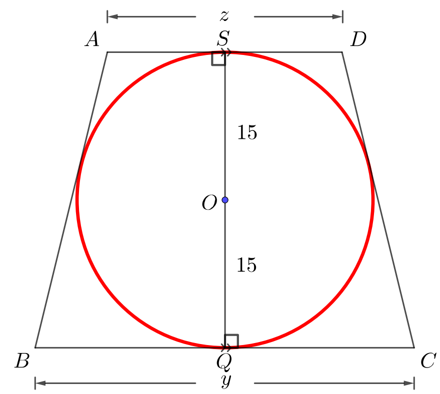
Solution

Solution 1

Let z represent the length of side AD and y represent the length of side BC.

Draw a line segment through O perpendicular to both AD and BC.

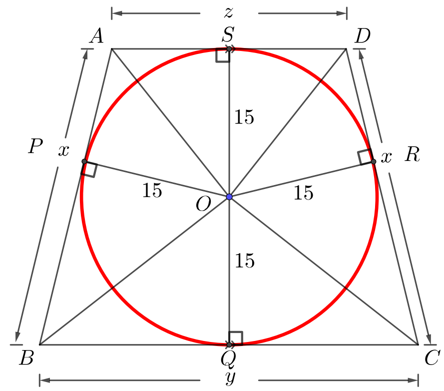
Using the second result given after the problem statement, we know that this perpendicular meets AD at the point of tangency S and BC at the point of tangency Q.



Both OS and OQ are radii of the circle. Since $SQ = OS + OQ$, it follows that SQ is a diameter of the circle and has length 30 cm. Since SQ is perpendicular to the two parallel sides of the trapezoid, we can use SQ as the height of the trapezoid.

Using the formula for the area of a trapezoid, we obtain

$$\begin{aligned}
\text{Area} &= \frac{SQ \times (AD + BC)}{2} \\
1000 &= \frac{30 \times (z + y)}{2} \\
1000 &= 15(z + y) \\
\frac{1000}{15} &= z + y \\
\frac{200}{3} &= z + y \quad (1)
\end{aligned}$$



Let the length of the two equal sides, AB and CD, be x.

Join the centre O to each of the vertices of $ABCD$, creating four triangles, $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle DOA$.

Connect the centre O to each of the points of tangency, P, Q, R, and S, on AB, BC, CD, and DA, respectively.

Each of these segments is a radius so $OP = OQ = OR = OS = 15$. From the first result given after the problem statement, we know that each of these segments is perpendicular to the tangent at the point of tangency so each of these radii can be a height of their respective triangles.

We can now find the area of the trapezoid a second way by summing the areas of the four triangles:

$$\text{Area } \triangle AOB = OP \times AB \div 2 = \frac{15x}{2}, \quad \text{Area } \triangle BOC = OQ \times BC \div 2 = \frac{15y}{2}$$

$$\text{Area } \triangle COD = OR \times CD \div 2 = \frac{15x}{2} \quad \text{Area } \triangle DOA = OS \times AD \div 2 = \frac{15z}{2}$$

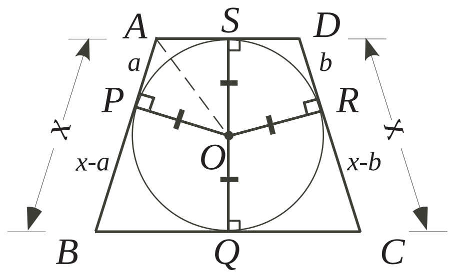
$$\begin{aligned}
\text{Area } \triangle AOB + \text{Area } \triangle BOC + \text{Area } \triangle COD + \text{Area } \triangle DOA &= 1000 \\
\frac{15x}{2} + \frac{15y}{2} + \frac{15x}{2} + \frac{15z}{2} &= 1000 \\
15x + \frac{15y}{2} + \frac{15z}{2} &= 1000 \\
15x + \frac{15y + 15z}{2} &= 1000 \\
15x + \frac{15(y+z)}{2} &= 1000 \\
\text{But } y + z = \frac{200}{3} \text{ from (1) above so} \qquad 15x + \frac{15(\frac{200}{3})}{2} &= 1000 \\
15x + 500 &= 1000 \\
15x &= 500 \\
x &= \frac{100}{3}
\end{aligned}$$

Therefore, the lengths of AB and DC are each $33\frac{1}{3}$ cm.

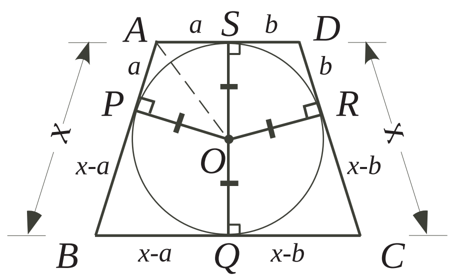
Solution 2

Let P, Q, R, and S be the points of tangency on AB, BC, CD, and DA, respectively. Draw OP, OQ, OR, and OS. Then $OP = OQ = OR = OS = 15$ since they are each radii of the circle. Also, since a line drawn from the centre of the circle to a point of tangency is perpendicular to the tangent, $\angle OPA = \angle OQB = \angle ORD = \angle OSA = 90°$.

Let $AB = x$, $AP = a$ and $DR = b$. Therefore, $DC = x$, $PB = x - a$ and $RC = x - b$. The following diagram shows all of the given and found information.



Join A to O forming two right triangles, $\triangle APO$ and $\triangle ASO$. Using the Pythagorean Theorem, $AP^2 = AO^2 - OP^2$ and $AS^2 = AO^2 - OS^2$. But $OP = OS$ since they are both radii. So the two expressions are equal and $AS = AP = a$ follows.
Using exactly the same reasoning that was used to show $AP = AS = a$, we can show $DR = DS = b$, $BP = BQ = x - a$ and $CR = CQ = x - b$. This new information has been added to the diagram below.



We can now use the area of a trapezoid formula. As was shown in Solution 1, $SQ = SO + OQ$ is a height of the trapezoid. Therefore,

$$\begin{aligned}
\text{Area Trapezoid } ABCD &= SQ \times (AD + BC) \div 2 \\
1000 &= (SO + OQ) \times ((AS + SD) + (BQ + QC)) \div 2 \\
1000 &= (15 + 15) \times ((a + b) + (x - a + x - b)) \div 2 \\
1000 &= (30) \times (2x) \div 2 \\
1000 &= 30x \\
\frac{100}{3} &= x
\end{aligned}$$

Therefore, the lengths of AB and DC are each $33\frac{1}{3}$ cm.

At the end of the statement of the problem, two facts were given. As an extension to this problem, the solver may wish to try to prove these two facts.

Problem of the Week

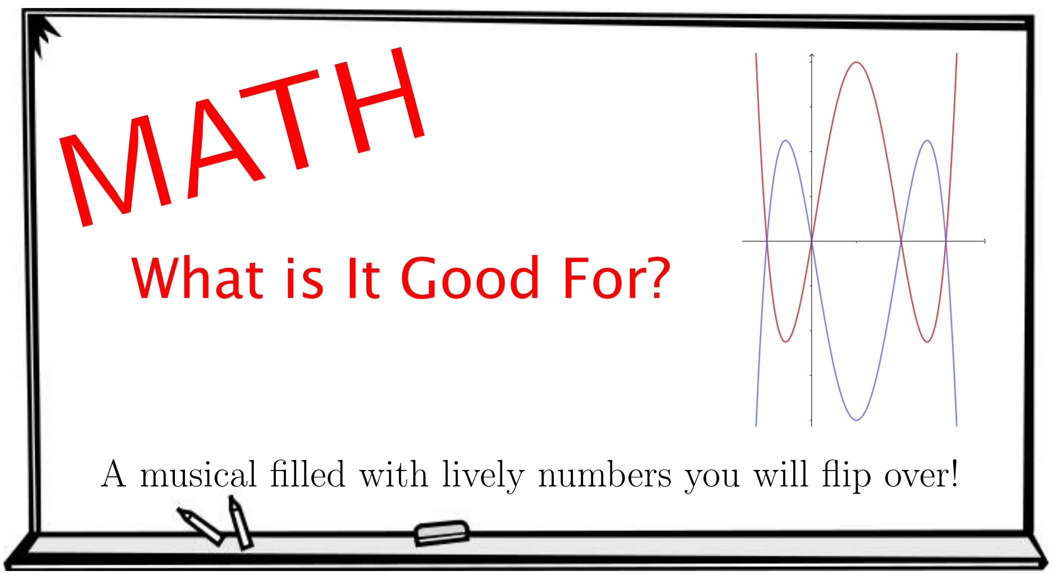
Problem D

Stick With It!

The junior and senior students at Mathville High School are going to present an exciting musical entitled, "Math, What is it Good For?". A large group of students came out to an information meeting. After a brief introduction to the musical, 15 senior students decided that it was not for them and they left. At that point, twice as many junior students as senior students remained.

Later in the meeting, after the 15 senior students had left, $\frac{3}{4}$ of the junior students and $\frac{1}{3}$ of the remaining senior students also left. This left 8 more senior students than junior students. All of the remaining students stuck it out and went on to produce an amazing product.

How many students remained to perform in the school musical, "Math, What is it Good For?"



MATH

Problem of the Week

Problem D and Solution

Stick With It!

What is It Good For?

Problem

The junior and senior students at Mathville High School are going to present an exciting musical entitled, "Math, What is it Good For?". A large group of students came out to an information meeting. After a brief introduction to the musical, 15 senior students decided that it was not for them and they left. At that point, twice as many junior students as senior students remained. Later in the meeting, after the 15 senior students had left, $\frac{3}{4}$ of the junior students and $\frac{1}{3}$ of the remaining senior students also left. This left 8 more senior students than junior students. All of the remaining students stuck it out and went on to produce an amazing product. How many students remained to perform in the school musical, "Math, What is it Good For?"

Solution

Let j represent the number of juniors and s represent the number of seniors present at the start of the information meeting.

After 15 seniors leave, $(s - 15)$ seniors remain. The number of juniors is twice the number of seniors at this point so $j = 2(s - 15)$ which simplifies to $j = 2s - 30$. (1)

Then $\frac{3}{4}$ of the juniors depart, leaving $\frac{1}{4}$ of the juniors or $\frac{1}{4}j$.
Also, $\frac{1}{3}$ of the seniors depart, leaving $\frac{2}{3}$ of the seniors remaining or $\frac{2}{3}(s - 15)$.
Now the number of seniors is 8 more than the number of juniors so $\frac{2}{3}(s - 15) = \frac{1}{4}j + 8$.
Multiplying by 12, the equation simplifies to $8(s - 15) = 3j + 96$. This further simplifies to $8s - 120 = 3j + 96$. (2)

At this point we can use substitution or elimination to solve the system of equations. In this case, we substitute (1) into (2) for j to solve for s.

$$\begin{aligned} 8s - 120 &= 3(2s - 30) + 96 \\ 8s - 120 &= 6s - 90 + 96 \\ 2s &= 126 \\ s &= 63 \end{aligned}$$

Substituting $s = 63$ into (1), $j = 2(63) - 30 = 126 - 30 = 96$.

The original number of students is $j + s = 96 + 63 = 159$.

The number of juniors remaining is $\frac{1}{4}j = \frac{1}{4}(96) = 24$.

The number of seniors remaining is $\frac{2}{3}(s - 15) = \frac{2}{3}(63 - 15) = \frac{2}{3}(48) = 32$.

The total number of students remaining to successfully perform in the musical is $24 + 32 = 56$.

The final production was an amazing success even though only 56 of the initial 159 students remained.

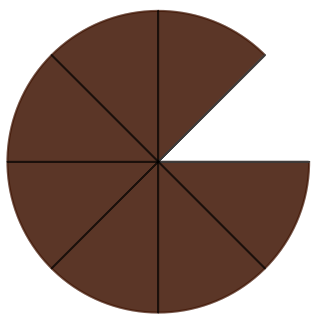
Problem of the Week
Problem D
More Cake Please

For Amanda's birthday, Rhett made an amazing, cylindrical, chocolate cream cheesecake. The radius and height of the cake were the same. Rhett cut the cake into 8 congruent slices and ate the first slice for quality control purposes.

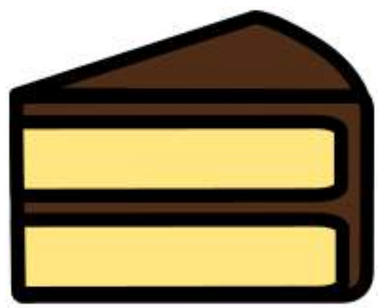
Rhett then posed the following two questions to Amanda:

a) After removing my slice, is there more or less total surface area (top, bottom and all exposed sides) in the remaining cake?

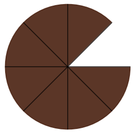
b) By what percentage, to 1 decimal place, has the remaining surface area increased or decreased?



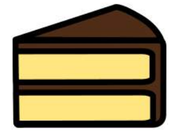
Top View with
Rhett's Slice Removed



Rhett's Slice



Problem of the Week
Problem D and Solution
More Cake Please

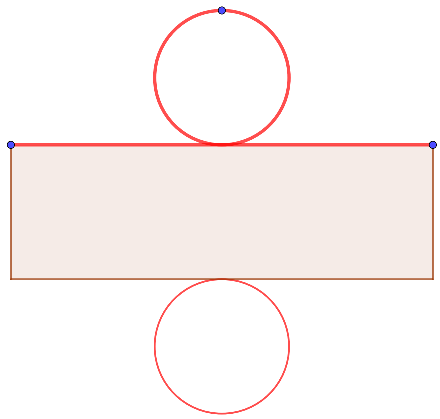


Problem

For Amanda's birthday, Rhett made an amazing, cylindrical, chocolate cream cheesecake. The radius and height of the cake were the same. Rhett cut the cake into 8 congruent slices and ate the first slice for quality control purposes. After removing Rhett's slice, is there more or less total surface area (top, bottom and all exposed sides) in the remaining cake, and by what percentage, to 1 decimal place, has the remaining surface area increased or decreased?

Solution

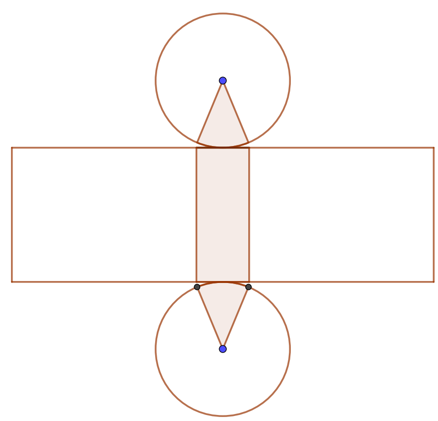
A net illustrating the 3 parts that make up the total surface area is shown to the right.



The total surface area includes the areas of two circles with radius r and a rectangle with length equal to the circumference of the circle and width equal to the height of the cake, $h = r$.

$$\begin{aligned}\text{Total Surface Area} &= 2\left(\pi r^2\right) + (2\pi r)(r)\\ &= 2\pi r^2 + 2\pi r^2\\ &= 4\pi r^2\end{aligned}$$

A net illustrating the 3 parts removed from the total surface area is shown to the right.

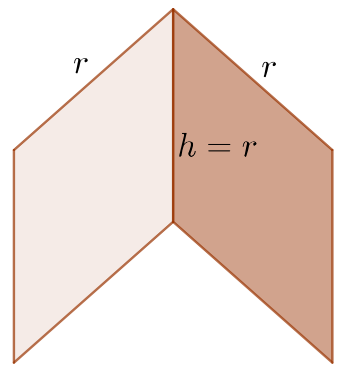


The surface area removed includes $\frac{1}{8}$ of the area of each of two circles with radius r and a rectangle $\frac{1}{8}$ of the area of the original rectangle area. Therefore, the surface area removed is $\frac{1}{8}$ of the total surface area.

$$\begin{aligned}\text{Surface Area Removed} &= \frac{1}{8}\left(4\pi r^2\right)\\ &= \frac{1}{2}\pi r^2\end{aligned}$$

But there are two areas that are added as a result of removing the slice.

A diagram illustrating the 2 parts added to the total surface area is shown to the right.



The surface area added includes 2 rectangles, each with length r and width $h = r$.

$$\begin{aligned}\text{Surface Area Added} &= 2(r)(r)\\ &= 2r^2\end{aligned}$$

We can now calculate the new total surface area.

$$\begin{aligned}\text{Surface Area} &= \text{Original Surface Area} - \text{Surface Area Removed} + \text{Surface Area Added}\\ &= 4\pi r^2 - \frac{1}{2}\left(\pi r^2\right) + 2r^2\\ &= 4\pi r^2 + r^2\left(-\frac{1}{2}\pi + 2\right)\end{aligned}$$

Now $\frac{1}{2}\pi < 2$, so $-\frac{1}{2}\pi + 2 > 0$, and the surface area actually increases as a result of removing the slice.

To calculate the percentage that the area has increased, divide the increase by the original area. The increase is $r^2\left(-\frac{1}{2}\pi + 2\right)$.

$$\begin{aligned}\text{Percentage Increase in Area} &= \frac{r^2\left(-\frac{1}{2}\pi + 2\right)}{4\pi r^2} \times 100\%\\ &= \frac{\left(-\frac{1}{2}\pi + 2\right)}{4\pi} \times 100\%\\ &= \left(-\frac{1}{8} + \frac{1}{2\pi}\right) \times 100\%\\ &= \left(\frac{-\pi + 4}{8\pi}\right) \times 100\%\\ &\approx 3.4\%\end{aligned}$$

The surface area of the cake increases by approximately 3.4% after the slice has been removed.

Problem of the Week
Problem D
Running Out of Digits

You initially have 100 of each digit from 0 to 9. This means you have 1000 digits in total. This count for each digit is shown in the table below.

Digit	0	1	2	3	4	5	6	7	8	9
# Remaining	100	100	100	100	100	100	100	100	100	100

Now start counting by ones, from 1. Each time you say a number you must remove the digits required to make the number from your stockpile of digits. For example, after you have counted from 1 to 13, the above table now looks like:

Digit	0	1	2	3	4	5	6	7	8	9
# Remaining	99	94	98	98	99	99	99	99	99	99

What is the largest number you can count to without running out of the digits needed to form the number?

0 1 2 3 4

5 6 7 8 9

0 1 2 3 4
5 6 7 8 9

Problem of the Week
Problem D and Solution
Running Out of Digits

Problem

You initially have 100 of each digit from 0 to 9. This means you have 1000 digits in total. Now start counting by ones, from 1. Each time you say a number you must remove the digits required to make the number from your stockpile of digits.
What is the largest number you can count to without running out of the digits needed to form the number?

Solution

In the integers 1 to 99, each digit, other than zero, will be used the same number of times. Zero will be used 9 less times than every other digit. As soon as you get to the the number 100, the 1s will be used more frequently. So, let's just count the number of 1s used until we run out.

From 1 to 99, a 1 is used in the units digit 10 times: $\{1, 11, 21, \cdots, 91\}$. There are 10 numbers with a tens digit of 1: $\{10, 11, 12, \cdots, 18, 19\}$. Therefore, by the time we reach 99, we have used $10 + 10 = 20$ ones.

From 100 to 109, we use 10 ones plus another in the number 101. This makes a total of 11 more ones, so we have now used $20 + 11 = 31$ ones.

From 110 to 119 we use 10 ones for hundreds digits and 10 ones for tens digits and one 1 for the units digit in 111. This makes a total of 21 more ones, so we have now used $31 + 21 = 52$ ones.

For 120 to 129, 130 to 139, 140 to 149, and 150 to 159 we use the same number of ones as we did counting from 100 to 109. So, to get from 120 to 159 we use $11 + 11 + 11 + 11 = 44$ ones, and have now used a total of $52 + 44 = 96$ ones.

Now we can count until we use up the 4 remaining ones: 160, 161 and 162.

When we try to count 163, there are no ones remaining.

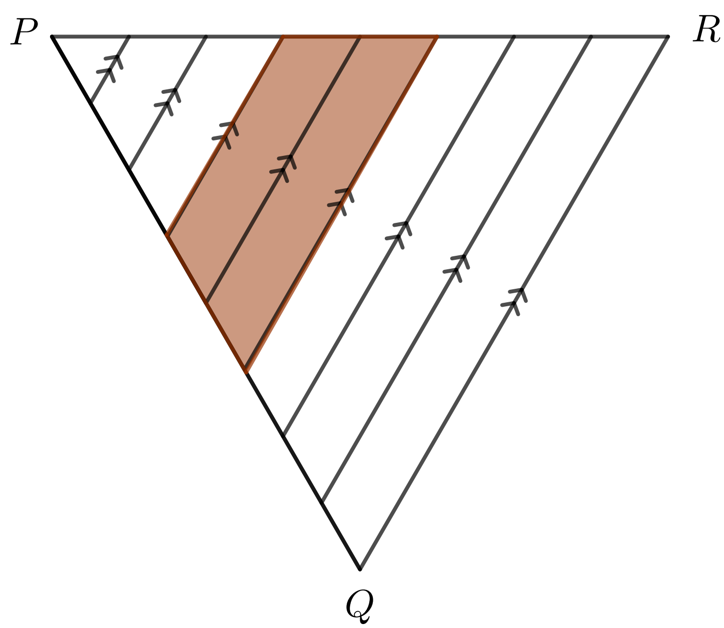
Therefore, the largest number we can count to before running out of the necessary digits to create the number is 162.

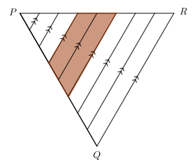
Problem of the Week

Problem D

Shady Area

$\triangle PQR$ is an equilateral triangle with sides of length 32 cm. Two sides of the triangle, PR and PQ, are each divided into 8 segments of equal length. Each point of division on PR is connected to its corresponding point of division on PQ, creating 7 line segments, as shown. Each of the new line segments is parallel to QR, the third side of the triangle. What is the area of the shaded trapezoid?



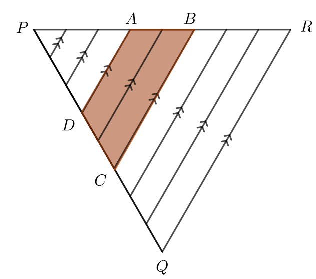


Problem of the Week
Problem D and Solution
Shady Area

Problem

$\triangle PQR$ is an equilateral triangle with sides of length 32 cm. Two sides of the triangle, PR and PQ, are each divided into 8 segments of equal length. Each point of division on PR is connected to its corresponding point of division on PQ, creating 7 line segments, as shown. Each of the new line segments is parallel to QR, the third side of the triangle. What is the area of the shaded trapezoid?

Solution
Solution 1:



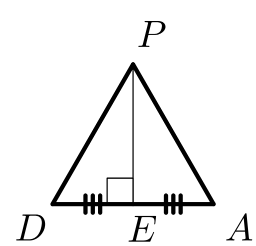
Label the vertices of the trapezoid A, B, C and D, as shown on the diagram. In this solution we will subtract the area of $\triangle PDA$ from the area of $\triangle PCB$ to find the area of trapezoid $ABCD$.

PR and PQ are divided into 8 equal segments, each of length $32 \div 8 = 4$ cm. PD and PA are each made up of 3 of the equal segments, and PC and PB are each made up of 5 of the equal segments. It follows that $PD = PA = 12$ cm and $PC = PB = 20$ cm.

First we will show that $\triangle PDA$ and $\triangle PCB$ are equilateral triangles.
Since $\triangle PQR$ is equilateral, $\angle PRQ = \angle PQR = \angle QPR = 60^\circ$. Since $\angle DPA$, $\angle CPB$ and $\angle QPR$ are the same angle, $\angle DPA = \angle CPD = \angle QPR = 60^\circ$.

Since $DA \parallel CB \parallel QR$, $\angle PDA = \angle PCB = \angle PQR = 60^\circ$ and $\angle PAD = \angle PBC = \angle PRQ = 60^\circ$.

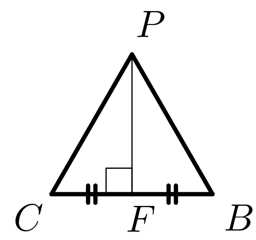
Since each angle in $\triangle PDA$ and $\triangle PCB$ is 60°, both triangles are equilateral. $\triangle PDA$ has side length 12 cm and $\triangle PBC$ has side length 20 cm.



In $\triangle PDA$, drop a perpendicular from P to E on DA. Since PDA is an equilateral triangle, E is the midpoint of DA and it follows that $DE = \frac{1}{2}DA = 6$ cm. Using the Pythagorean Theorem, $PE^2 = PD^2 - DE^2 = 12^2 - 6^2 = 108$ and $DE = \sqrt{108} = \sqrt{36 \times 3} = 6\sqrt{3}$ cm (since $DE \geq 0$).

The area of $\triangle PDA = \frac{(PE)(DA)}{2} = \frac{(6\sqrt{3})(12)}{2} = 36\sqrt{3}$ cm^2.

In $\triangle PCB$, drop a perpendicular from P to F on CB. Since PCB is an equilateral triangle, F is the midpoint of CB and it follows that $CF = \frac{1}{2}CB = 10$ cm. Using the Pythagorean Theorem, $PF^2 = PC^2 - CF^2 = 20^2 - 10^2 = 300$ and $PF = \sqrt{300} = \sqrt{100 \times 3} = 10\sqrt{3}$ cm (since $PF \geq 0$).

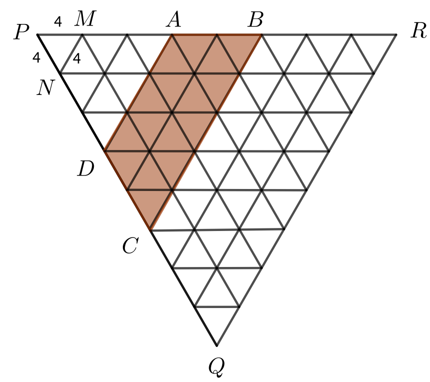


The area of $\triangle PCB = \frac{(PF)(CB)}{2} = \frac{(10\sqrt{3})(20)}{2} = 100\sqrt{3}$ cm^2.
The area of trapezoid $ABCD$ = area $\triangle PCB$ − area $\triangle PDA = 100\sqrt{3} - 36\sqrt{3} = 64\sqrt{3}$ cm^2.

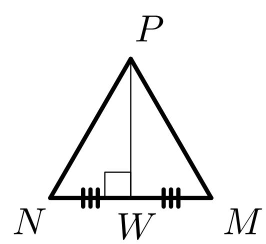
(Note that we could also have found the lengths of PE and PF by recognizing that $\triangle PDE$ and $\triangle PCF$ are 30°- 60°- 90° triangles with sides in the ratio $1 : \sqrt{3} : 2$.)

Solution 2:

As seen in the diagram to the right, we can tile $\triangle PQR$ with the top left equilateral triangle of side length 4 cm. Three equilateral triangles fit in the first trapezoid from the left, 5 equilateral triangles fit in the second trapezoid from the left, 7 equilateral triangles fit in the third trapezoid from the left, 9 equilateral triangles fit in the fourth trapezoid from the left, and so on. The shaded region contains 16 equilateral triangles, each of which has side length 4 cm. To find the area of the shaded region, we will find the area of an equilateral triangle with side length 4 cm, and multiply the result by 16.



Let the small equilateral triangle be $\triangle PNM$. In $\triangle PNM$, drop a perpendicular from P to W on NM. Since PNM is an equilateral triangle, W is the midpoint of NM and it follows that $NW = \frac{1}{2}NM = 2$ cm.
Using the Pythagorean Theorem, $PW^2 = PN^2 - NW^2 = 4^2 - 2^2 = 12$ and $PW = \sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$ cm (since $PW \geq 0$).
The area of $\triangle PNM = \frac{(PW)(NM)}{2} = \frac{(2\sqrt{3})(4)}{2} = 4\sqrt{3}$ cm^2.



Therefore, the area of the shaded region is $16 \times 4\sqrt{3} = 64\sqrt{3}$ cm^2.

(Note that we could also have found the length of PW by recognizing that $\triangle PNW$ is a 30°- 60°- 90° triangle with sides in the ratio $1 : \sqrt{3} : 2$.)

Problem of the Week

Problem D

Counting Lights, Not Sheep

Some nights it is difficult to get to sleep. On one such night, John counted the number of LEDs (Light-Emitting Diodes) on his clock radio that were on to make each individual time from 10:00 PM to 12:59 AM. He did this instead of counting sheep. When it was 11:11, for example, he noted that 8 of the LEDs were on.

During the time that John was awake from 10:00 PM to 12:59 AM, how many of the times had exactly 20 of the LEDs on?

Here is some information about John's clock radio:

- Only times from 12:00 to 11:59 can be displayed.
- Each digit is made up of seven LEDs which are turned off or on depending on the particular digit to be displayed. The digit 2 has five of the seven LEDs on while the digit 8 has all seven LEDs on. All of the digits are shown in the diagram below.
- For times from 10:00 to 12:59, all four digits are used.
- For times from 1:00 to 9:59, only three digits are displayed. The leftmost digit is completely off for these times.

Problem of the Week
Problem D and Solution
Counting Lights, Not Sheep

Problem

Some nights it is difficult to get to sleep. On one such night, John counted the number of LEDs (Light-Emitting Diodes) that were on to make each individual time from 10:00 PM to 12:59 AM. He did this instead of counting sheep. When it was 11:11, for example, he noted that 8 of the LEDs were on. During the time that John was awake from 10:00 PM to 12:59 AM, how many of the times had exactly 20 of the LEDs on?

Solution

First, we will create a chart showing the digit and the number of LEDs which are on when that particular digit is displayed.

Digit	1	2	3	4	5	6	7	8	9	0
# of LEDs ON	2	5	5	4	5	6	3	7	6	6

Since John is awake from 10:00 PM to 12:59 AM, the leftmost digit must be a 1. During the time he is awake, 2 of the LEDs on the leftmost digit are on meaning that 18 LEDs must be on for the remaining three digits. The second digit will be either a 0, 1 or 2. We will consider the possibilities for these three cases.

1. Times from 10:00 to 10:59

   The first two digits are 10, and a total of $2 + 6$ or 8 LEDs are on. The last two digits must be from 00 to 59, inclusive, such that the total number of LEDs on between the two digits is $20 - 8 = 12$. Using the chart above, we can form valid digit combinations so that 12 LEDs are on. This can be done by using two digits, each with 6 LEDs on, or by using one digit with 5 LEDs on and another digit with 7 LEDs on. The digit combination must form a valid time, so for this last possibility to work, an 8 must be used and it can only be used as the final digit. The only valid combinations are 00, 06, 09, 28, 38, and 58. Therefore, at 10:00, 10:06, 10:09, 10:28, 10:38, and 10:58 there are 20 LEDs on.

2. Times from 11:00 to 11:59

   The first two digits are 11, and a total of $2 + 2 = 4$ LEDs are on. The remaining two digits must have a total of 16 LEDs on between them. Each digit can have a maximum of 7 LEDs on, so it is not possible for the remaining two digits to have a total of 16 LEDs on. There are no times between 11:00 and 11:59 that have 20 LEDs on.

3. Times from 12:00 to 12:59

   The first two digits are 12, and a total of $2 + 5 = 7$ LEDs are on. The remaining two digits must have a total of 13 LEDs on between them. The only way for this to happen is if one of the digits has 6 LEDs on and the other has 7 LEDs on. The only digit with 7 LEDs on is the 8. If there is a valid time between 12:00 and 12:59, the last digit must be an 8. Then the only choices for the third digit are 6, 9 and 0. Of these three possibilities, only 0 can be used to form a valid time. Therefore, the only time between 12:00 and 12:59 with 20 LEDs on is 12:08.

Combining the three cases, there are a total of $6 + 0 + 1 = 7$ times in which exactly 20 of the LEDs are on. The only times from 10:00 PM to 12:59 AM which have exactly 20 LEDs on are 10:00 PM, 10:06 PM, 10:09 PM, 10:28 PM, 10:38 PM, 10:58 PM and 12:08 AM.

For Further Thought:

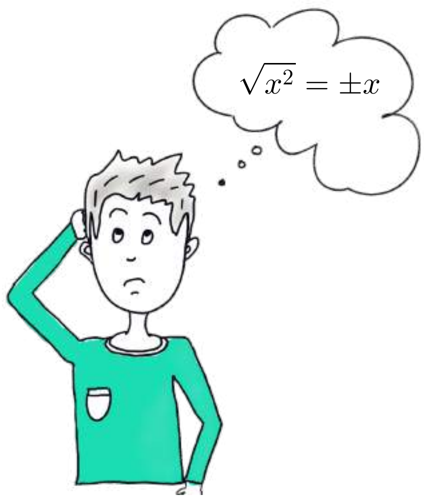
Ignoring AM and PM, how many times from 1:00 to 9:59 use exactly 20 LEDs to display the time? The answer may surprise you.

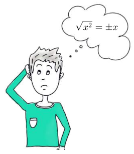
Problem of the Week
Problem D
What's that Total?

We know the following about the numbers a, b and c:

$$(a+b)^2 = 9,\ (b+c)^2 = 25, \text{ and } (a+c)^2 = 81.$$

If $a+b+c \geq 1$, determine the **number** of possible values for $a+b+c$.





Problem of the Week

Problem D and Solution

What's that Total?

Problem

We know the following about the numbers a, b and c:

$$(a+b)^2 = 9,\ (b+c)^2 = 25, \text{ and } (a+c)^2 = 81.$$

If $a+b+c \geq 1$, determine the **number** of possible values for $a+b+c$.

Solution

Since $(a+b)^2 = 9$, $a+b = \pm 3$. Since $(b+c)^2 = 25$, $b+c = \pm 5$. And since $(a+c)^2 = 81$, $a+c = \pm 9$.

Now $(a+b)+(b+c)+(a+c) = 2a+2b+2c = 2(a+b+c)$. This quantity is two times the value of the quantity we are looking for.

The following chart summarizes all possible combinations of values for $a+b$, $b+c$, and $a+c$ and the resulting values of $2a+2b+2c$ and $a+b+c$. The final column of the chart states a yes or no answer to whether the value of $a+b+c$ is ≥ 1.

$a+b$	$b+c$	$a+c$	$2a+2b+2c$	$a+b+c$	$a+b+c \geq 1$? (yes / no)
3	5	9	17	8.5	yes
3	5	-9	-1	-0.5	no
3	-5	9	7	3.5	yes
3	-5	-9	-11	-5.5	no
-3	5	9	11	5.5	yes
-3	5	-9	-7	-3.5	no
-3	-5	9	1	0.5	no
-3	-5	-9	-17	-8.5	no

Therefore, there are three possible values of $a+b+c$ such that $a+b+c \geq 1$.

It should be noted that for each of the three possibilities, values for a, b, and c which produce each value can be determined but that was not the question asked.

Problem of the Week

Problem D

Mind Your Ps and Qs and Rs

Three digit numbers PQR, QQP and PQQ are formed using the single digits P, Q and R such that:

$$PQR + 2 = PQQ$$

AND

$$PQR \times 2 = QQP$$

Determine all possible combinations of values of P, Q and R that satisfy the two equations.

Problem of the Week
Problem D and Solution
Mind Your Ps and Qs and Rs

Problem

The letters P, Q, and R represent single digits. Determine all possible combinations of values of P, Q and R, given that: $PQR + 2 = PQQ$ and $PQR \times 2 = QQP$

Solution

From $PQR + 2 = PQQ$, we note that the number PQQ is 2 more than the number PQR and that the first two digits of each number are the same. From this, $Q = R + 2$ follows. We can prove this using place value. The number $PQR = 100 \times P + 10 \times Q + R = 100P + 10Q + R$. The number $PQQ = 100 \times P + 10 \times Q + Q = 100P + 10Q + Q$.
Then,

$$\begin{aligned}(100P + 10Q + R) + 2 &= 100P + 10Q + Q\\ 100P + 10Q + R + 2 &= 100P + 11Q\\ R + 2 &= Q, \text{ as above.}\end{aligned}$$

From this expression we can obtain a restriction on the possible integer values of R. R must be an integer from 0 to 7, inclusive. If $R = 8$, then $R = Q + 2 = 10$ and Q is not a single digit.

From here, we will show two possible solutions.

<u>Solution 1</u>

Representing $PQR \times 2 = QQP$ using place value,

$$\begin{aligned}(100P + 10Q + R) \times 2 &= 100Q + 10Q + P\\ 200P + 20Q + 2R &= 110Q + P\\ 199P &= 90Q - 2R\\ \text{Substituting } R+2 \text{ for } Q: \quad 199P &= 90(R + 2) - 2R\\ 199P &= 90R + 180 - 2R\\ P &= \frac{88R + 180}{199}\end{aligned}$$

We are looking for an integer value of R from 0 to 7 such that $88R + 180$ is a multiple of 199. The only value of R that produces a multiple of 199 when substituted into $88R + 180$ is $R = 7$. When $R = 7$, $P = \frac{88R+180}{199} = \frac{88(7)+180}{199} = 4$ and $Q = R + 2 = 9$.

Therefore, the only values that satisfy the system of equations are $P = 4$, $Q = 9$ and $R = 7$.

We can verify this result:

$$PQR + 2 = 497 + 2 = 499 = PQQ \text{ and}$$

$$PQR \times 2 = 497 \times 2 = 994 = QQP.$$

Solution 2

If R must be an integer from 0 to 7, inclusive, then Q must be an integer from 2 to 9, inclusive.
From the second equation, we also know that P would be the units digit of $2R$.
We can now use a table to determine all possible values for P, Q and R

Q	R	$2R$	P	PQR	QQP	Does $2 \times PQR = QQP$?
2	0	0	0	020	220	No
3	1	2	2	231	332	No
4	2	4	4	442	444	No
5	3	6	6	653	556	No
6	4	8	8	864	668	No
7	5	10	0	075	770	No
8	6	12	2	286	882	No
9	7	14	4	497	994	Yes

We will now need to verify that the final row also works for $PQR + 2 = PQQ$.

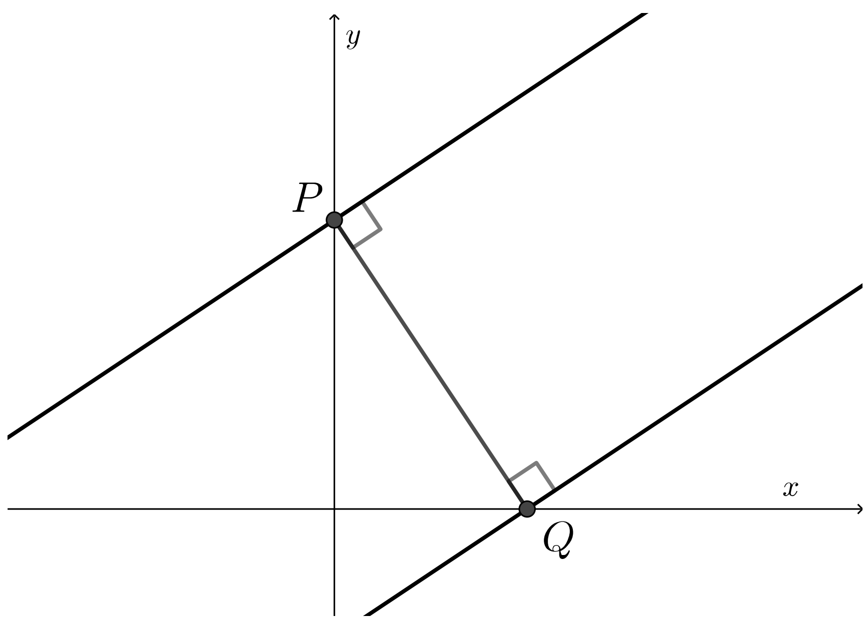
$$PQR + 2 = 497 + 2 = 499 = PQQ.$$

Therefore, the only values that satisfy the system of equations are $P = 4$, $Q = 9$ and $R = 7$.

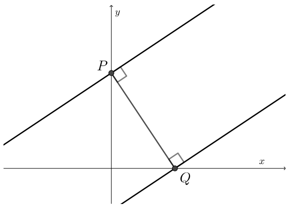
Problem of the Week
Problem D
Crossing Points in General

Two distinct lines are drawn such that the first line passes through point P on the y-axis and the second line passes through point Q on the x-axis. Line segment PQ is perpendicular to both lines.

If the line through P has equation $y = mx + k$, then determine the y-intercept of the line through Q in terms of m and k.



Suggestion: If you are finding the general problem difficult to start, consider first solving a problem with a specific example for the line through P, like $y = 4x + 3$, and then attempt the more general problem.



Problem of the Week

Problem D and Solution

Crossing Points in General

Problem

Two distinct lines are drawn such that the first line passes through point P on the y-axis and the second line passes through point Q on the x-axis. Line segment PQ is perpendicular to both lines. If the line through P has equation $y = mx + k$, then determine the y-intercept of line through Q in terms of m and k.

Solution

For ease of reference, we will call the first line l_1 and the second line l_2.

Let b represent the y-intercept of l_2.

Since l_1 has equation $y = mx + k$, we know that the slope of l_1 is m and the y-intercept is k. Therefore, P is the point $(0, k)$.

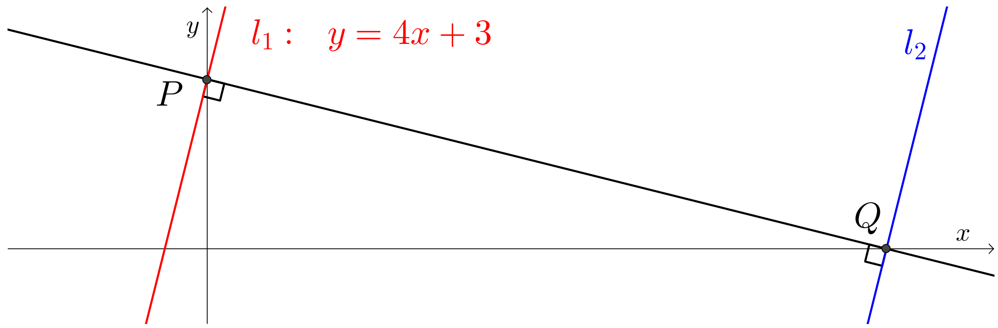
Since PQ is perpendicular to both lines, it follows that l_1 is parallel to l_2. Also, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, $\text{slope}(PQ) = -\frac{1}{m}$. Since k is the y-intercept of the perpendicular segment PQ and the slope of PQ is $-\frac{1}{m}$, the equation of the line through PQ is $y = -\frac{1}{m}x + k$.

To find the x-intercept of $y = -\frac{1}{m}x + k$, set $y = 0$ and solve for x. If $y = 0$, then $0 = -\frac{1}{m}x + k$ and $\frac{1}{m}x = k$. The result $x = mk$ follows. Therefore, the x-intercept of $y = -\frac{1}{m}x + k$ is mk and the coordinates of Q are $(mk, 0)$.

We can now find the y-intercept of l_2 since we know $Q(mk, 0)$ is on l_2 and the slope of l_2 is m. Substituting into the slope-intercept form of the line, $y = mx + b$, we obtain $0 = (m)(mk) + b$ which simplifies to $b = -m^2k$.

Therefore, the y-intercept of l_2, the line through Q, is $-m^2k$.

For the student who solved the problem using $y = 4x + 3$ as the equation of l_1, you should have obtained the answer -48 for the y-intercept of l_2, the line through Q. A full solution to this problem is provided on the next page.



Let l_1 represent the line $y = 4x + 3$. Let l_2 represent the second line, the line through Q.

From the equation of l_1 we know that the slope is 4 and the y-intercept is 3. Therefore P is the point (0,3).

Since $PQ \perp l_1$ and $PQ \perp l_2$, it follows that $l_1 \parallel l_2$. Also, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, slope$(PQ) = -\frac{1}{4}$. Since 3 is the y-intercept of PQ and the slope of PQ is $-\frac{1}{4}$, the equation of the line through PQ is $y = -\frac{1}{4}x + 3$.

The x-intercepts of the line through perpendicular PQ and the line l_2 are the same since both lines intersect at Q on the x-axis. To find this x-intercept, set $y = 0$ in $y = -\frac{1}{4}x + 3$. Then $0 = -\frac{1}{4}x + 3$ and $\frac{1}{4}x = 3$. The result, $x = 12$, follows. The x-intercept of the line through perpendicular PQ and the line l_2 is 12 and point Q is $(12, 0)$.

We can now find equation of l_2 since $Q(12, 0)$ is on l_2 and the slope of l_2 is 4. Substituting $x = 12$, $y = 0$ and $m = 4$ into $y = mx + b$, we obtain $0 = (4)(12) + b$ which simplifies to $b = -48$. The equation of l_2 is $y = 4x - 48$ and the y-intercept is -48.

This is the same result we obtained from the general solution on the previous page.

Data Management (D)

TAKE ME TO THE COVER

Problem of the Week
Problem D
Go for the Gold

For orientation days at the University of Waterloo, various activities are available for incoming students to participate in. One activity is a simple game in which students reach into a box and randomly select a golf ball. The golf balls are all identical in size and are either black or gold. If a student selects a gold golf ball, then they win a prize. After a golf ball is selected, it is returned to the box.

Initially, the box contained 300 gold golf balls and students had a 1 in 5 chance of selecting a gold golf ball. The organizers want to increase the chances of selecting a gold golf ball to 3 in 10. In order to do this, they add complete packages of golf balls. Each package of golf balls contains 60 golf balls, of which 65% are black.

How many full packages of golf balls must be added to the box?

gold

Problem of the Week
Problem D and Solution
Go for the Gold

black

Problem

For orientation days at the University of Waterloo, various activities are available for incoming students to participate in. One activity is a simple game in which students reach into a box and randomly select a golf ball. The golf balls are all identical in size and are either black or gold. If a student selects a gold golf ball, then they win a prize. After a golf ball is selected, it is returned to the box. Initially, the box contained 300 gold golf balls and students had a 1 in 5 chance of selecting a gold golf ball. The organizers want to increase the chances of selecting a gold golf ball to 3 in 10. In order to do this, they add complete packages of golf balls. Each package of golf balls contains 60 golf balls, of which 65% are black. How many full packages of golf balls must be added to the box?

Solution

There were initially 300 gold golf balls in the box and the students had a 1 in 5 chance of winning. This means that there was is 4 in 5 chance of selecting a black golf ball. Therefore, there were four times as many black golf balls as gold golf balls. That is, there were $4 \times 300 = 1200$ black golf balls and a total of $300 + 1200 = 1500$ golf balls in the box before any new packages were added.

Each new package of golf balls contains 60 golf balls. Since 65% are black, there are $0.65 \times 60 = 39$ black golf balls and $60 - 39 = 21$ gold golf balls in each new package.

Let n represent the number of packages of golf balls added to the box to increase the chances of winning from 1 in 5 to 3 in 10. By adding n packages of golf balls to the box, we are adding $21n$ gold golf balls and $60n$ golf balls to the box. After adding the packages, the box will contain $300 + 21n$ gold golf balls and $1500 + 60n$ golf balls. Since the chances of winning are now 3 in 10,

$$\begin{aligned}
\frac{\text{the number of gold golf balls}}{\text{the total number of golf balls}} &= \frac{3}{10}\\
\frac{300+21n}{1500+60n} &= \frac{3}{10}\\
10(300+21n) &= 3(1500+60n)\\
3000+210n &= 4500+180n\\
30n &= 1500\\
n &= 50
\end{aligned}$$

Therefore, 50 new packages of golf balls must be added to the box to raise the chances of winning from 1 in 5 to 3 in 10.

We can check this easily. After adding 50 new packages of golf balls, there would be $300 + 21(50) = 1350$ gold golf balls and a total of $1500 + 60(50) = 4500$ golf balls. Then, the ratio of gold golf balls to the total number of golf balls in the box is $\frac{1350}{4500} = \frac{3}{10}$, as required.

Problem of the Week

Problem D

The Number You Have Reached is ...

Maryam remembers a few things about a friend's cell phone number but cannot completely remember all ten digits. Maryam is absolutely certain that the first seven digits of the number are (122) 578 8.

Maryam does remember a few interesting things about the number. With regard to the numeric keypad, she remembers the following things:

- Any digit in the phone number that is different from the previous digit somehow touches the next digit of the phone number. For example, the digit 1 on the keypad touches digits 2, 4 and 5. The digit 5 on the keypad touches every digit but 0.
- The phone number contains three distinct pairs of repeating consecutive digits, but three digits in a row are never the same. (Of the numbers Maryam remembers, there are already two distinct pairs of repeating digits, so the third pair cannot be 22 or 88.)

1	2	3
4	5	6
7	8	9
*	0	#

Maryam's phone plan has unlimited free local calling. She will try different combinations until the friend is reached. How many different phone numbers could Maryam possibly end up trying until the correct number is reached?

Problem of the Week
Problem D and Solution
The Number You Have Reached is ...

Problem

Maryam remembers a few things about a friend's cell phone number but cannot completely remember all ten digits. Maryam is absolutely certain that the first seven digits of the number are (122) 578 8. Maryam does remember a few interesting things about the number. With regard to the numeric keypad, she remembers the following: any digit in the phone number that is different from the previous digit somehow touches the next digit of the phone number and the phone number contains three distinct pairs of repeating consecutive digits but three digits in a row are never the same.

1	2	3
4	5	6
7	8	9
*	0	#

Maryam's phone plan has unlimited free local calling. She will try different combinations until the friend is reached. How many different phone numbers could Maryam possibly end up trying until the correct number is reached?

Solution

Six numbers on the keypad are adjacent to the 8. The first missing digit cannot be an 8 since there are never three consecutive digits the same. So, the only possibilities for the first missing digit are 4, 5, 6, 7, 9 and 0. We will examine each case.

1. The first missing digit is a 4.

If the digit after the 4 is also a 4, we have the third pair of repeating digits. The final digit could be any of the five digits adjacent to the 4 on the keypad but it cannot be a 4. There are 5 possibilities for phone numbers in which the first two missing digits are 4's.

If the digit after the 4 is not a 4, then the final two digits must be the same but cannot be 2's or 8's. There are then 3 possibilities for the final two digits, 11, 55 and 77.

Therefore, if the first missing digit is a 4, then there are $5 + 3 = 8$ possible phone numbers.

2. The first missing digit is a 5.

If the digit after the 5 is also a 5, we have the third pair of repeating digits. The final digit could be any of the eight digits adjacent to the 5 on the keypad but it cannot be a 5. There are 8 possibilities for phone numbers in which the first two missing digits are 5's.

If the digit after the 5 is not a 5, then the final two digits must be the same but cannot be 2's or 8's. There are then 6 possibilities for the final two digits, 11, 33, 44, 66, 77 and 99.

Therefore, if the first missing digit is a 5, then there are $8 + 6 = 14$ possible phone numbers.

3. The first missing digit is a 6.

If the digit after the 6 is also a 6, we have the third pair of repeating digits. The final digit could be any of the five digits adjacent to the 6 on the keypad but it cannot be a 6. There are 5 possibilities for phone numbers in which the first two missing digits are 6's.

If the digit after the 6 is not a 6, then the final two digits must be the same but cannot be 2's or 8's. There are then 3 possibilities for the final two digits, 33, 55 and 99.

Therefore, if the first missing digit is a 6, then there are $5 + 3 = 8$ possible phone numbers.

4. The first missing digit is a 7.

If the digit after the 7 is also a 7, we have the third pair of repeating digits. The final digit could be any of the four digits adjacent to the 7 on the keypad but it cannot be a 7. There are 4 possibilities for phone numbers in which the first two missing digits are 7's.

If the digit after the 7 is not a 7, then the final two digits must be the same but cannot be 2's or 8's. There are then 3 possibilities for the final two digits, 44, 55 and 00.

Therefore, if the first missing digit is a 7, then there are $4 + 3 = 7$ possible phone numbers.

5. The first missing digit is a 9.

If the digit after the 9 is also a 9, we have the third pair of repeating digits. The final digit could be any of the four digits adjacent to the 9 on the keypad but it cannot be a 9. There are 4 possibilities for phone numbers in which the first two missing digits are 9's.

If the digit after the 9 is not a 9, then the final two digits must be the same but cannot be 2's or 8's. There are then 3 possibilities for the final two digits, 55, 66 and 00.

Therefore, if the first missing digit is a 9, then there are $4 + 3 = 7$ possible phone numbers.

6. The first missing digit is a 0.

If the digit after the 0 is also a 0, we have the third pair of repeating digits. The final digit could be any of the three digits adjacent to the 0 on the keypad but it cannot be a 0. There are 3 possibilities for phone numbers in which the first two missing digits are zeroes.

If the digit after the 0 is not a 0, then the final two digits must be the same but cannot be 2's or 8's. There are then 2 possibilities for the final two digits, 77 and 99.

Therefore, if the first missing digit is a 0, then there are $3 + 2 = 5$ possible phone numbers.

By adding the number of possibilities from each distinct case we can determine the total number of possible phone numbers. There are $8 + 14 + 8 + 7 + 7 + 5 = 49$ possible phone numbers to try in order for Maryam to be able to get the friend's correct number.

Problem of the Week

Problem D

This is the Year

The positive integers can be arranged as follows.

Row 1	1					
Row 2	2	3				
Row 3	4	5	6			
Row 4	7	8	9	10		
Row 5	11	12	13	14	15	
Row 6	16	17	18	19	20	21
$\vdots$						

More rows and columns continue to list the positive integers in order, with each new row containing one more integer than the previous row.

How many integers less than 2020 are in the *column* that contains the number 2020?

Did you know that the sum of the positive integers from 1 to n can be determined using the formula $\frac{n(n+1)}{2}$? That is, $1+2+3+4+\ldots+(n-1)+n=\frac{n(n+1)}{2}$. For example, the sum of the integers $1+2+3+4=\frac{4(5)}{2}=10$. This result can be verified by simply adding the 4 numbers. You can also easily verify that the sum of the first 5 positive integers is $\frac{5(6)}{2}=15$.

This formula may be useful in solving this problem. As an extension, one may wish to prove this formula holds for any positive integer n.

Problem of the Week
Problem D and Solution
This is the Year

Problem
The positive integers can be arranged as follows.

Row 1	1					
Row 2	2	3				
Row 3	4	5	6			
Row 4	7	8	9	10		
Row 5	11	12	13	14	15	
Row 6	16	17	18	19	20	21
⋮						

More rows and columns continue to list the positive integers in order, with each new row containing one more integer than the previous row. How many integers less than 2020 are in the *column* that contains the number 2020?

Solution

In the table given, there is one number in Row 1, there are two numbers in Row 2, three numbers in Row 3, and so on, with n numbers in Row n.
The numbers in the rows list the positive integers in order beginning at 1 in Row 1, with each new row containing one more integer than the previous row. Thus, the last number in each row is equal to the sum of the number of numbers in each row of the table up to that row.

For example, the last number in Row 4 is 10, which is equal to the sum of the number of numbers in rows 1, 2, 3, and 4. But the number of numbers in each row is equal to the row number. So 10 is equal to the sum $1 + 2 + 3 + 4$.
That is, the last number in Row n is equal to the sum

$$1 + 2 + 3 + 4 + \ldots + (n - 1) + n = \frac{n(n+1)}{2}.$$

To find out which row the integer 2020 occurs in, we could use organized trial and error.
Using trial and error, we find that since $\frac{63(63+1)}{2} = 2016$, then the last number in Row 63 is 2016.
We then find $\frac{64(64+1)}{2} = 2080$, so the last number in Row 64 is 2080.
Since 2020 is between 2016 and 2080, then it must appear somewhere in the 64th row.

Alternatively, we could determine the row the integer 2020 occurs in by using the

quadratic formula to solve the equation $\frac{n(n+1)}{2} = 2020$ for n.

$$\begin{aligned}
\frac{n(n+1)}{2} &= 2020 \\
n(n+1) &= 4040 \\
n^2 + n - 4040 &= 0 \\
n &\approx 63.1 \text{ (using the quadratic formula and } n \geq 0\text{)}
\end{aligned}$$

This means that the integer 2020 will be in Row 64.

If we look at the original arrangement given, the first number in Row 6, which is 16, has five numbers in the column above it. The second number in Row 6, which is 17, has four numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.
Also, note that the first number in Row 5, which is 11, has four numbers in the column above it. The second number in Row 5, which is 12, has three numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.
The pattern is that the first number in Row n has $n - 1$ numbers in the column above it. The second number in Row n has $n - 2$ numbers in the column above it. For every other entry in the row, it has one fewer number of numbers in the column above it than the entry before it in the row.

Therefore, the first number of Row 64, which is 2017, will have 63 numbers in the column above it. The second number of Row 64, which is 2018, will have 62 numbers in the column above it. The third number of Row 64, which is 2019, will have 61 numbers in the column above it. The fourth number of Row 64, which is 2020, will have 60 numbers in the column above it.

Therefore, there are 60 integers less than 2020 in the column that contains the number 2020.

Problem of the Week

Problem D

Tokens Taken

Three bags each contain tokens. The green bag contains 22 round green tokens, each with a different integer from 1 to 22. The red bag contains 15 triangular red tokens, each with a different integer from 1 to 15. The blue bag contains 10 square blue tokens, each with a different integer from 1 to 10.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $22 \times 15 \times 10 = 3300$ different combinations of tokens created by selecting one token from each bag. Note that selecting the 7 red token, the 5 blue token and 3 green token is different than selecting the 5 red token, 7 blue token and the 3 green token. The order of selection does not matter.

You select one token from each bag. What is the probability that two or more of the selected tokens have the number 5 on them?

Problem of the Week
Problem D and Solution
Tokens Taken

Problem

Three bags each contain tokens. The green bag contains 22 round green tokens, each with a different integer from 1 to 22. The red bag contains 15 triangular red tokens, each with a different integer from 1 to 15. The blue bag contains 10 square blue tokens, each with a different integer from 1 to 10.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $22 \times 15 \times 10 = 3300$ different combinations of tokens created by selecting one token from each bag. Note that selecting the 7 red token, the 5 blue token and 3 green token is different than selecting the 5 red token, 7 blue token and the 3 green token. The order of selection does not matter.

You select one token from each bag. What is the probability that two or more of the selected tokens have the number 5 on them?

Solution

Solution 1

There are 22 different numbers which can be chosen from the green bag, 15 different numbers which can be chosen from the red bag, and 10 different numbers which can be chosen from the blue bag. So there are a total of $22 \times 15 \times 10 = 3300$ different combinations of numbers which can be produced by selecting one token from each bag.

To count the number of possibilities for a 5 to appear on at least two of the tokens, we will consider cases.

1. Each of the selected tokens has a 5 on it.

   This can only occur in 1 way.

2. A 5 appears on the green token and on the red token but not on the blue token.

   There are 9 choices for the blue token excluding the 5. A 5 can appear on the green token and on the red token but not on the blue token in 9 ways.

3. A 5 appears on the green token and on the blue token but not on the red token.

   There are 14 choices for the red token excluding the 5. A 5 can appear on the green token and on the blue token but not on the red token in 14 ways.

4. A 5 appears on the red token and on the blue token but not on the green token.

   There are 21 choices for the green token excluding the 5. A 5 can appear on the red token and on the blue token but not on the green token in 21 ways.

Summing the results from each of the cases, the total number of ways for a 5 to appear on at least two of the tokens is $1 + 9 + 14 + 21 = 45$. The probability of 5 appearing on at least two of the tokens is $\frac{45}{3300} = \frac{3}{220}$.

Solution 2

This solution uses a known result from probability theory. If the probability of event A occurring is a, the probability of event B occurring is b, the probability of event C occurring is c, and the results are not dependent on each other, then the probability of all three events happening is $a \times b \times c$.

The probability of a specific number being selected from the green bag is $\frac{1}{22}$ and the probability of any specific number not being selected from the green bag is $\frac{21}{22}$.

The probability of a specific number being selected from the red bag is $\frac{1}{15}$ and the probability of any specific number not being selected from the red bag is $\frac{14}{15}$.

The probability of a specific number being selected from the blue bag is $\frac{1}{10}$ and the probability of any specific number not being selected from the blue bag is $\frac{9}{10}$.

In the following we will use $P(p, q, r)$ to mean the probability of p being selected from the green bag, q being selected from the red bag, and r being selected from the blue bag. So, $P(5, 5,\ \text{not } 5)$ means that we want the probability of a 5 being selected from the green bag, a 5 being selected from the red bag, and anything but a 5 being selected from the blue bag.

$$\begin{aligned}
& \text{Probability of 5 being selected from at least two of the bags} \\
= {} & \text{Probability of 5 from each bag} + \text{Probability of 5 from exactly 2 bags} \\
= {} & \text{P(5, 5, 5)} + \text{P(5, 5, not 5)} + \text{P(5, not 5, 5)} + \text{P(not 5, 5, 5)} \\
= {} & \frac{1}{22} \times \frac{1}{15} \times \frac{1}{10} + \frac{1}{22} \times \frac{1}{15} \times \frac{9}{10} + \frac{1}{22} \times \frac{14}{15} \times \frac{1}{10} + \frac{21}{22} \times \frac{1}{15} \times \frac{1}{10} \\
= {} & \frac{1}{3300} + \frac{9}{3300} + \frac{14}{3300} + \frac{21}{3300} \\
= {} & \frac{45}{3300} \\
= {} & \frac{3}{220}
\end{aligned}$$

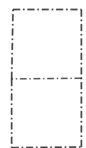
The probability of 5 appearing on at least two of the tokens is $\dfrac{3}{220}$.

Computational Thinking (C)

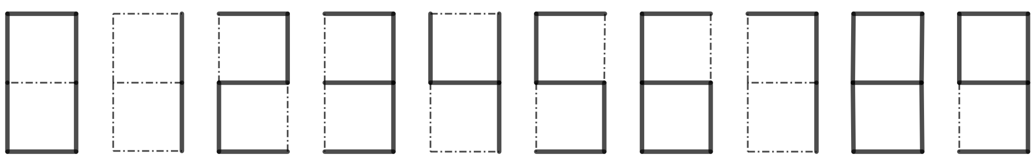
TAKE ME TO THE COVER

Problem of the Week
Problem D
Seven Segments or Less

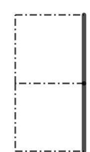
A 7-segment display consists of seven LEDs arranged in a rectangular fashion, as shown below. Each of the seven LEDs is called a segment because, when illuminated, the segment forms part of a numerical digit to be displayed. On a digital clock, each digit can be formed by lighting up some of the seven segments of the following 7-segment display.



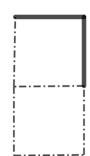
Below, each of the digits from 0 to 9 are shown by lighting up some of those seven segments. For example, the digit 8 uses all seven segments, but the digit 1 uses only the two right vertical segments.



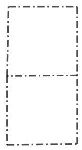
If a segment burns out, there could be a problem distinguishing which digit is showing. For example, if the top segment is burnt out then the display to the right could still be the digit 1 or it could be the digit 7.



However, if the bottom right vertical segment is the only segment burnt out, then we can unambiguously determine that the digit on the right must be the digit 7.



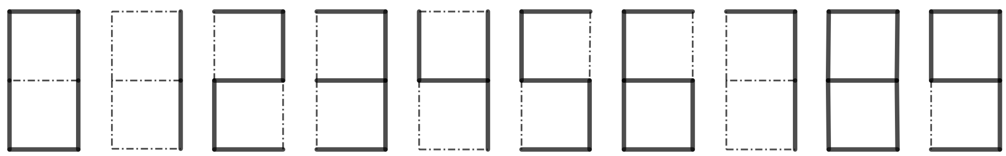
What is the fewest number of working segments that are needed so that each digit can be unambiguously determined?



Problem of the Week
Problem D and Solution
Seven Segments or Less

Problem

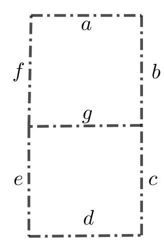
A 7-segment display consists of seven LEDs arranged in a rectangular fashion, as shown above. Each of the seven LEDs is called a segment because, when illuminated, the segment forms part of a numerical digit to be displayed. On a digital clock, each digit can be formed by lighting up some of the seven segments on a 7-segment display. Below, each of the digits from 0 to 9 are shown by lighting up some of those seven segments.



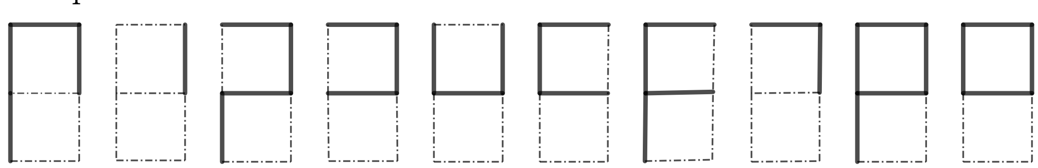
What is the fewest number of working segments that are needed so that each digit can be unambiguously determined?

Solution

Consider the following labelling of the 7 segments with the letters a through g:



Segment a must work to distinguish between digits 1 and 7.
Segment b must work to distinguish between digits 6 and 8.
Segment e must work to distinguish between digits 5 and 6 and also between the digits 8 and 9.
Segment f must work to distinguish between digits 3 and 9.
Segment g must work to distinguish between digits 0 and 8.
If we do not have c and d working, we would have the following 10 digits, each of which is unique:



Therefore, the fewest number of working segments needed is 5.

The Beaver Computing Challenge (BCC):

This problem is based on a previous BCC problem. The BCC is designed to get students with little or no previous experience excited about computing. Questions are inspired by topics in computer science and connections to Computer Science are described in the solutions to all past BCC problems. If you enjoyed this problem, you may want to explore the BCC contest further.

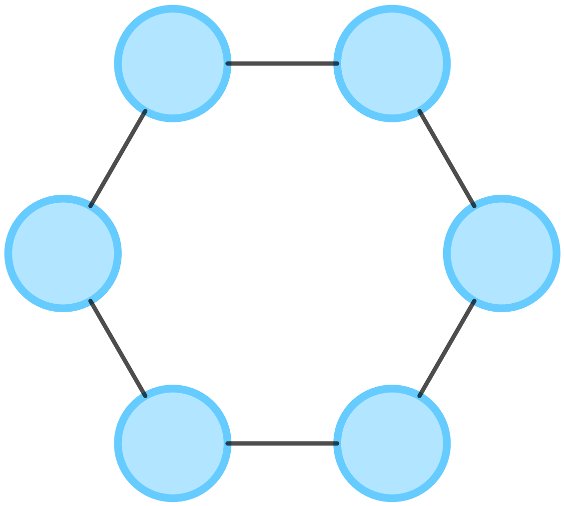
Connections to Computer Science:

Pattern recognition algorithms are algorithms with take in complex information, such as a picture or sound, and try to categorize it. For example, pattern recognition algorithms are used to detect facial features, such as eyes, mouth, and nose, for security and social media applications. These algorithms look for identifying information and attempt to provide a "most likely" answer, using statistical models.

Pattern recognition is a branch of a larger area of computer science called machine learning, that focuses on the recognition of patterns and regularities in data. One of the approaches to recognition is to extract specific features, that allow uniquely identified objects. This task focuses on identifying these key features to distinguish between each digit.

Problem of the Week
Problem D
Initial This

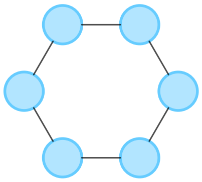
In this two-player game, the vertices of a regular hexagon are each covered by a circle.



On a turn, a player may either initial one circle or initial two adjacent circles. (The two adjacent circles must be directly connected by an outside edge of the hexagon.) Players alternate turns. The player initialing the last circle or last pair of adjacent circles is the winner.

Two players, Cameron and Dale, play the game with Cameron going first.
One of the players, Cameron or Dale, can always win the game. Describe which player can always win and the winning strategy for that player.

You will probably better understand this game by playing it a few times with a partner. Be sure to take turns going first.



Problem of the Week

Problem D and Solution

Initial This

Problem

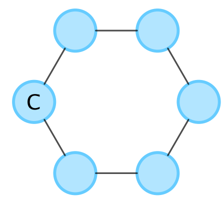
In this two-player game, the vertices of a regular hexagon are each covered by a circle. On a turn, a player may either initial one circle or initial two adjacent circles. (The two adjacent circles must be directly connected by an outside edge of the hexagon.) Players alternate turns. The player initialing the last circle or last pair of adjacent circles is the winner. Two players, Cameron and Dale, play the game with Cameron going first. One of the players, Cameron or Dale, can always win the game. Describe which player can always win and the winning strategy for that player.

Solution

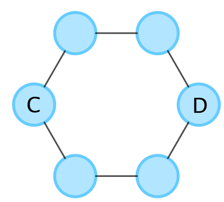
There are two types of possible first moves for Cameron.

(a) Cameron could initial exactly one blank circle.

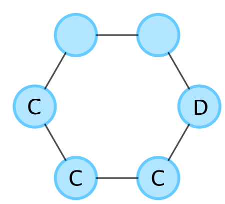
(b) Cameron could initial two adjacent blank circles.



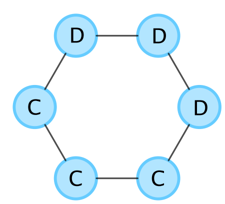
Let us look at (a) first. We can rotate the hexagon without changing the game, so we will assume that Cameron initials the circle on the left with a ‘C’. This is shown on the diagram to the right.



In this case, Dale should initial the circle on the opposite side of the hexagon with a ‘D’ as shown in the second diagram to the right.

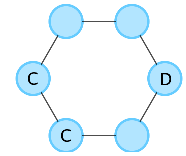


On his second turn, Cameron may initial two adjacent circles, either at the top or bottom. This is shown on the diagram to the right.

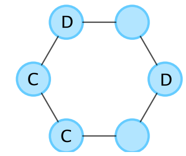


Dale should then initial the two remaining adjacent circles on the opposite side to win the game. This is shown on the diagram to the right.

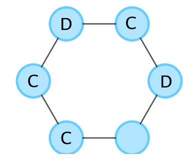
Instead, on his second turn, Cameron may initial only one circle. A possible selection is shown to the right.



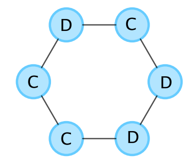
Dale should then initial one of the two adjacent unmarked circles on the other side of the hexagon. A possible selection is shown to the right.



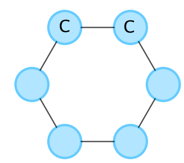
Now there are two non-adjacent unmarked circles left. On his third turn, Cameron must initial one of the unmarked circles.



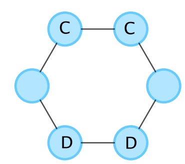
Dale then wins the game by initialing the last unmarked circle.



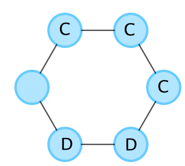
What happens in situation (b) where Cameron initials two adjacent unmarked circles? Again, we can rotate the hexagon without changing the game, so we will assume that Cameron initials the two unmarked circles at the top.



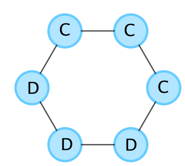
In this case, Dale should mark the two adjacent unmarked circles on the bottom.



As in the first situation, there are two unmarked circles left and they are not adjacent, so Cameron must initial exactly one of the unmarked circles.



Dale then initials the last unmarked circle to win the game.



We have considered all of the possible cases and have shown that Dale has a winning strategy. Dale should "copy" Cameron by initialing the same number of unmarked circles as Cameron but on the opposite side of the hexagon. If Dale follows this strategy, Dale will always win.

For Further Thought

Without changing the rules of the game, how would the strategy change if instead of a hexagon with 6 circles, there were a heptagon with 7 circles?

Problem of the Week

Problem D

Counting Lights, Not Sheep

Some nights it is difficult to get to sleep. On one such night, John counted the number of LEDs (Light-Emitting Diodes) on his clock radio that were on to make each individual time from 10:00 PM to 12:59 AM. He did this instead of counting sheep. When it was 11:11, for example, he noted that 8 of the LEDs were on.

During the time that John was awake from 10:00 PM to 12:59 AM, how many of the times had exactly 20 of the LEDs on?

Here is some information about John's clock radio:

- Only times from 12:00 to 11:59 can be displayed.
- Each digit is made up of seven LEDs which are turned off or on depending on the particular digit to be displayed. The digit 2 has five of the seven LEDs on while the digit 8 has all seven LEDs on. All of the digits are shown in the diagram below.
- For times from 10:00 to 12:59, all four digits are used.
- For times from 1:00 to 9:59, only three digits are displayed. The leftmost digit is completely off for these times.

Problem of the Week
Problem D and Solution
Counting Lights, Not Sheep

Problem

Some nights it is difficult to get to sleep. On one such night, John counted the number of LEDs (Light-Emitting Diodes) that were on to make each individual time from 10:00 PM to 12:59 AM. He did this instead of counting sheep. When it was 11:11, for example, he noted that 8 of the LEDs were on. During the time that John was awake from 10:00 PM to 12:59 AM, how many of the times had exactly 20 of the LEDs on?

Solution

First, we will create a chart showing the digit and the number of LEDs which are on when that particular digit is displayed.

Digit	1	2	3	4	5	6	7	8	9	0
# of LEDs ON	2	5	5	4	5	6	3	7	6	6

Since John is awake from 10:00 PM to 12:59 AM, the leftmost digit must be a 1. During the time he is awake, 2 of the LEDs on the leftmost digit are on meaning that 18 LEDs must be on for the remaining three digits. The second digit will be either a 0, 1 or 2. We will consider the possibilities for these three cases.

1. Times from 10:00 to 10:59

   The first two digits are 10, and a total of $2 + 6$ or 8 LEDs are on. The last two digits must be from 00 to 59, inclusive, such that the total number of LEDs on between the two digits is $20 - 8 = 12$. Using the chart above, we can form valid digit combinations so that 12 LEDs are on. This can be done by using two digits, each with 6 LEDs on, or by using one digit with 5 LEDs on and another digit with 7 LEDs on. The digit combination must form a valid time, so for this last possibility to work, an 8 must be used and it can only be used as the final digit. The only valid combinations are 00, 06, 09, 28, 38, and 58. Therefore, at 10:00, 10:06, 10:09, 10:28, 10:38, and 10:58 there are 20 LEDs on.

2. Times from 11:00 to 11:59

   The first two digits are 11, and a total of $2 + 2 = 4$ LEDs are on. The remaining two digits must have a total of 16 LEDs on between them. Each digit can have a maximum of 7 LEDs on, so it is not possible for the remaining two digits to have a total of 16 LEDs on. There are no times between 11:00 and 11:59 that have 20 LEDs on.

3. Times from 12:00 to 12:59

   The first two digits are 12, and a total of $2 + 5 = 7$ LEDs are on. The remaining two digits must have a total of 13 LEDs on between them. The only way for this to happen is if one of the digits has 6 LEDs on and the other has 7 LEDs on. The only digit with 7 LEDs on is the 8. If there is a valid time between 12:00 and 12:59, the last digit must be an 8. Then the only choices for the third digit are 6, 9 and 0. Of these three possibilities, only 0 can be used to form a valid time. Therefore, the only time between 12:00 and 12:59 with 20 LEDs on is 12:08.

Combining the three cases, there are a total of $6 + 0 + 1 = 7$ times in which exactly 20 of the LEDs are on. The only times from 10:00 PM to 12:59 AM which have exactly 20 LEDs on are 10:00 PM, 10:06 PM, 10:09 PM, 10:28 PM, 10:38 PM, 10:58 PM and 12:08 AM.

For Further Thought:

Ignoring AM and PM, how many times from 1:00 to 9:59 use exactly 20 LEDs to display the time? The answer may surprise you.

Problem of the Week
Problem D
Winter Carnival Event

The Brrrr-well Winter Carnival has fun for all ages. Daisy, Delilah, Donny, and Duke have each decided to participate in the snowperson building event.

In decorating their snowperson, each of them has chosen a different colour of scarf from purple, red, green, or blue. In addition they have chosen one accessory from a top hat, earmuffs, flower, or carrot nose.

Daisy, Delilah, Donny, and Duke have all chosen to give their snowpeople coal buttons. However, each person has decided to give their snowperson a different number of buttons. One snowperson has 2 buttons, one has 3 buttons, one has 4 buttons and one has 5 buttons.

Using the following clues, determine the combination of accessories and buttons that each person used to create their snowperson.

1. The snowperson built by Daisy, who is wearing the top hat, has one fewer button than the snowperson wearing the red scarf, but one more button than the snowperson with the carrot nose.
2. The snowperson wearing the blue scarf, who is also wearing earmuffs, has two fewer buttons than the snowperson wearing the green scarf.
3. The snowperson wearing the purple scarf has one more button than the snowperson built by Delilah.
4. The snowperson built by Duke is wearing the flower.

You may find the following table useful in organizing your solution to this problem.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy												
Delilah												
Donny												
Duke												
Top Hat												
Earmuffs												
Flower												
Carrot Nose												
2 Buttons												
3 Buttons												
4 Buttons												
5 Buttons												

Problem of the Week
Problem D and Solution
Winter Carnival Event

Problem

The Brrrr-well Winter Carnival has fun for all ages. Daisy, Delilah, Donny, and Duke have each decided to participate in the snowperson building event. In decorating their snowperson, each of them has chosen a different colour of scarf from purple, red, green, or blue. In addition they have chosen one accessory from a top hat, earmuffs, flower, or carrot nose. Daisy, Delilah, Donny, and Duke have all chosen to give their snowpeople coal buttons. However, each person has decided to give their snowperson a different number of buttons. One snowperson has 2 buttons, one has 3 buttons, one has 4 buttons and one has 5 buttons. Using the following clues, determine the combination of accessories and buttons that each person used to create their snowperson.

1. The snowperson built by Daisy, who is wearing the top hat, has one fewer button than the snowperson wearing the red scarf, but one more button than the snowperson with the carrot nose.
2. The snowperson wearing the blue scarf, who is also wearing earmuffs, has two fewer buttons than the snowperson wearing the green scarf.
3. The snowperson wearing the purple scarf has one more button than the snowperson built by Delilah.
4. The snowperson built by Duke is wearing the flower.

Answer

We will present the answer first for those who want to check their work. The solution that follows represents one possible approach to arriving at a correct set of conclusions.

- Daisy's snowperson is wearing the green scarf and the top hat. It has four buttons.
- Delilah's snowperson is wearing the blue scarf and earmuffs. It has two buttons.
- Donny's snowperson is wearing the purple scarf and has the carrot nose. It has three buttons.
- Duke's snowperson is wearing the red scarf and flower. It has five buttons.

Solution

In our solution, we will we go through each clue and update the table based on the information in the clue. We will put an X in a cell if the combination indicated by the row and column for that cell is not possible, or a ✓ if it must be true.

From clue (1), we know that the snowperson built by Daisy is wearing the top hat. Thus, we can put a ✓ in the corresponding cell. We can also determine that the snowperson built by Daisy does not have 2 buttons or 5 buttons since it has one fewer button than the snowperson wearing the red scarf (excluding it from having 5 buttons) and one more button than the snowperson wearing the carrot nose (excluding it from having 2 buttons). We can place an X in each of the corresponding cells.

We can also place an X in the cells corresponding to the snowperson wearing a red scarf having 2 buttons and the snowperson who has a carrot nose having 5 buttons.

We can further conclude that the snowperson built by Daisy is not wearing the red scarf and does not have a carrot nose, and that the snowperson wearing the red scarf does not have a carrot nose. We can place an X in each of the corresponding cells. The table is updated to the right.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy		X			X			X	✓			X
Delilah												
Donny												
Duke												
Top Hat												
Earmuffs												
Flower												
Carrot Nose		X						X				
2 Buttons		X										
3 Buttons												
4 Buttons												
5 Buttons												

Since no two snowmen can have the same accessory, we know that the snowmen built by Delilah, Donny and Duke cannot be wearing the top hat. Similarly, since a snowperson cannot be wearing more than one accessory, we can determine that the snowperson built by Daisy cannot be wearing the earmuffs or the flower. We can place an X in each of the corresponding cells.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy		X			X			X	✓	X	X	X
Delilah									X			
Donny									X			
Duke									X			
Top Hat		X			X			X				
Earmuffs												
Flower												
Carrot Nose		X						X				
2 Buttons		X										
3 Buttons												
4 Buttons												
5 Buttons												

We know that the snowperson built by Daisy is wearing the top hat, and that the snowperson built by Daisy cannot have 2 buttons or 5 buttons, so we can conclude that the snowperson wearing the top hat also cannot have 2 buttons or 5 buttons. Similarly we can conclude that the snowperson wearing the top hat cannot be wearing the red scarf. We can place an X in each of the corresponding cells. The table is updated above.

From clue (2), we know that the snowperson wearing the blue scarf is also wearing earmuffs. Thus, we can put a ✓ in the corresponding cell.

We can also determine that the snowperson wearing the blue scarf cannot have 4 buttons or 5 buttons because he has two fewer than the snowperson wearing the green scarf. We can place an X in each of the corresponding cells.

We can also place an X in the cells corresponding to the snowperson wearing the green scarf having 2 buttons or 3 buttons. The table is updated to the right.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy		X			X			X	✓	X	X	X
Delilah									X			
Donny									X			
Duke									X			
Top Hat		X			X			X				
Earmuffs				✓								
Flower												
Carrot Nose		X						X				
2 Buttons		X	X									
3 Buttons			X									
4 Buttons				X								
5 Buttons				X								

Since no two snowmen can be wearing the same accessory or have the same colour of scarf, we can place an X in the cells corresponding to the snowmen wearing the purple, red or green scarf also wearing earmuffs. We can also place an X in the cells corresponding to the snowperson wearing the blue scarf also wearing the top hat, flower, or carrot nose since he cannot have more than one accessory.

Since we know that the snowperson wearing the blue scarf is wearing the earmuffs and that the snowperson wearing the blue scarf cannot have 4 buttons or 5 buttons, we know that the snowperson wearing the earmuffs cannot have 4 buttons or 5 buttons either. Therefore, we can place an X in each of the corresponding cells. The table is updated as shown to the right.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy		X			X			X	✓	X	X	X
Delilah									X			
Donny									X			
Duke									X			
Top Hat		X		X	X			X				
Earmuffs	X	X	X	✓			X	X				
Flower				X								
Carrot Nose		X		X				X				
2 Buttons		X	X									
3 Buttons			X									
4 Buttons				X								
5 Buttons				X								

We see that the snowperson wearing the red scarf is also wearing the flower and that the snowperson wearing the flower has 5 buttons. Thus, the snowperson wearing the red scarf has 5 buttons. We can add the corresponding ✓'s and X's. The table is updated as shown to the right.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy		X			X			X	✓	X	X	X
Delilah									X			
Donny									X			
Duke									X			
Top Hat		X		X	X			X				
Earmuffs	X	X	X	✓			X	X				
Flower	X	✓	X	X	X	X	X	✓				
Carrot Nose		X		X				X				
2 Buttons		X	X									
3 Buttons		X	X									
4 Buttons		X		X								
5 Buttons	X	✓	X	X								

We can see that the snowperson wearing the green scarf has 4 buttons and can add the corresponding ✓'s and X's. The table is updated as follows.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy		X			X			X	✓	X	X	X
Delilah									X			
Donny									X			
Duke									X			
Top Hat		X		X	X			X				
Earmuffs	X	X	X	✓			X	X				
Flower	X	✓	X	X	X	X	X	✓				
Carrot Nose		X		X				X				
2 Buttons		X	X									
3 Buttons		X	X									
4 Buttons	X	X	✓	X								
5 Buttons	X	✓	X	X								

From clue (3) we can determine that the snowperson wearing the purple scarf cannot have 2 buttons and that the snowperson built by Delilah cannot have 5 buttons. We can place an X in the corresponding cells. We can also place an X in the cell corresponding to the snowperson built by Delilah wearing the purple scarf. The table is updated as shown to the right.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy		X			X			X	✓	X	X	X
Delilah	X							X	X			
Donny									X			
Duke									X			
Top Hat		X		X	X			X				
Earmuffs	X	X	X	✓			X	X				
Flower	X	✓	X	X	X	X	X	✓				
Carrot Nose		X		X				X				
2 Buttons	X	X	X									
3 Buttons		X	X									
4 Buttons	X	X	✓	X								
5 Buttons	X	✓	X	X								

We can see that the snowperson wearing the blue scarf has 2 buttons and that the snowperson wearing the purple scarf has 3 buttons. We can add the corresponding ✓'s and X's. Then, since the snowperson wearing the blue scarf is also wearing earmuffs and has 2 buttons, we can conclude that the snowperson wearing earmuffs also has 2 buttons and can add the corresponding ✓'s and X's. The table is updated as shown to the right.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy		X			X			X	✓	X	X	X
Delilah	X							X	X			
Donny									X			
Duke									X			
Top Hat		X		X	X			X				
Earmuffs	X	X	X	✓	✓	X	X	X				
Flower	X	✓	X	X	X	X	X	✓				
Carrot Nose		X		X	X			X				
2 Buttons	X	X	X	✓								
3 Buttons	✓	X	X	X								
4 Buttons	X	X	✓	X								
5 Buttons	X	✓	X	X								

From clue (4) we know that the snowperson built by Duke is wearing the flower. We can add the corresponding ✓'s and X's. Then since we also know that the snowperson wearing the flower has 5 buttons and is wearing the red scarf, we can conclude that the snowperson built by Duke also has 5 buttons and is wearing the red scarf. We can add the corresponding ✓'s and X's, and the table is updated to the right.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy		X			X			X	✓	X	X	X
Delilah	X	X						X	X		X	
Donny		X						X	X		X	
Duke	X	✓	X	X	X	X	X	✓	X	X	✓	X
Top Hat		X		X	X			X				
Earmuffs	X	X	X	✓	✓	X	X	X				
Flower	X	✓	X	X	X	X	X	✓				
Carrot Nose		X		X	X			X				
2 Buttons	X	X	X	✓								
3 Buttons	✓	X	X	X								
4 Buttons	X	X	✓	X								
5 Buttons	X	✓	X	X								

We have now gone through all of the clues and it may appear as though we are stuck here. We need to analyze further and go back over the given information. In particular, let's go back to clue (1). From this clue we know that the snowperson built by Daisy has one fewer button than snowperson wearing the red scarf. Since the snowperson wearing the red scarf has 5 buttons, then the snowperson built by Daisy must have 4 buttons. We can add the corresponding ✓'s and X's.

We also know that the snowperson built by Daisy is wearing the top hat, so we can conclude that the snowperson wearing the top hat has 4 buttons. Notice also that the snowperson with 4 buttons is wearing the green scarf so we can conclude that the snowperson built by Daisy and the snowperson wearing the top hat are also wearing the green scarf. We can add the corresponding ✓'s and X's and the table is updated as follows.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy	X	X	✓	X	X	X	✓	X	✓	X	X	X
Delilah	X	X	X				X	X	X		X	
Donny		X	X				X	X	X		X	
Duke	X	✓	X	X	X	X	X	✓	X	X	✓	X
Top Hat	X	X	✓	X	X	X	✓	X				
Earmuffs	X	X	X	✓	✓	X	X	X				
Flower	X	✓	X	X	X	X	X	✓				
Carrot Nose		X	X	X	X		X	X				
2 Buttons	X	X	X	✓								
3 Buttons	✓	X	X	X								
4 Buttons	X	X	✓	X								
5 Buttons	X	✓	X	X								

We know now that the snowperson built by Donny is wearing the purple scarf and that the snowperson wearing the purple scarf has the carrot nose. Therefore, the snowperson built by Donny has the carrot nose. We can also see that the snowperson with the carrot nose has three buttons, therefore the snowperson built by Donny has 3 buttons. So, the snowperson built by Delilah has 2 buttons, is wearing the blue scarf, and is wearing the earmuffs. The table is updated as follows.

	Purple	Red	Green	Blue	2 Buttons	3 Buttons	4 Buttons	5 Buttons	Top Hat	Earmuffs	Flower	Carrot Nose
Daisy	X	X	✓	X	X	X	✓	X	✓	X	X	X
Delilah	X	X	X	✓	✓	X	X	X	X	✓	X	X
Donny	✓	X	X	X	X	✓	X	X	X	X	X	✓
Duke	X	✓	X	X	X	X	X	✓	X	X	✓	X
Top Hat	X	X	✓	X	X	X	✓	X				
Earmuffs	X	X	X	✓	✓	X	X	X				
Flower	X	✓	X	X	X	X	X	✓				
Carrot Nose	✓	X	X	X	X	✓	X	X				
2 Buttons	X	X	X	✓								
3 Buttons	✓	X	X	X								
4 Buttons	X	X	✓	X								
5 Buttons	X	✓	X	X								

The table is complete and we can summarize our results as follows:

The snowperson built by Daisy is wearing the green scarf, has 4 buttons, and is wearing the top hat.

The snowperson built by Delilah is wearing the blue scarf, has 2 buttons, and is wearing the earmuffs.

The snowperson built by Donny is wearing the purple scarf, has 3 buttons, and has the carrot nose.

The snowperson built by Duke is wearing the red scarf, has 5 buttons, and is wearing the flower.

Problem of the Week
Problem D
A De-Light-Ful Machine

A machine has 2020 lights and 1 button. Each button press changes the state of exactly 3 of the lights. That means if the light is currently on, it turns off, and if the light is currently off, it turns on. Before each button press, the user selects which 3 lights will change their state.

To begin with, all the lights on the machine are off. What is the fewest number of button presses required in order for all the lights to be on?

Hint: Start by thinking about a machine with fewer lights.

Problem of the Week
Problem D and Solution
A De-Light-Ful Machine

Problem

A machine has 2020 lights and 1 button. Each button press changes the state of exactly 3 of the lights. That means if the light is currently on, it turns off, and if the light is currently off, it turns on. Before each button press, the user selects which 3 lights will change their state. To begin with, all the lights on the machine are off. What is the fewest number of button presses required in order for all the lights to be on?

Hint: Start by thinking about a machine with fewer lights.

Solution

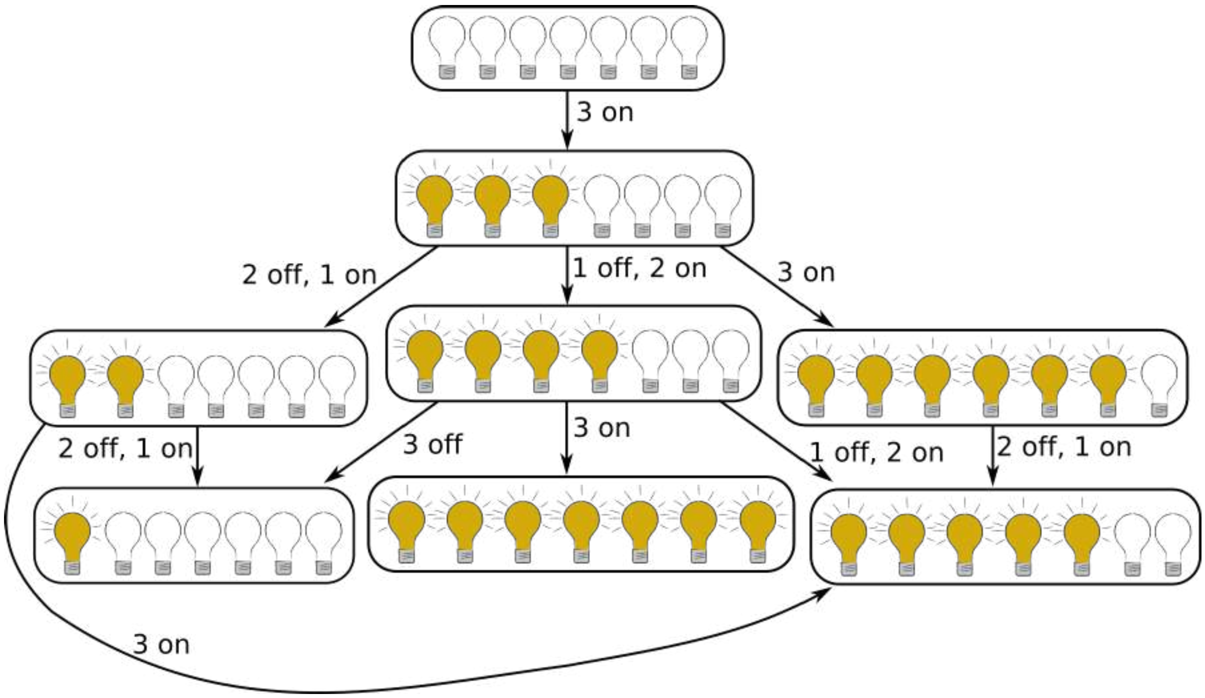
To turn on all the lights with the fewest number of button presses, we should turn on 3 lights with each button press, and not turn any lights off.

- The first button press would turn on 3 lights.
- The second button press would turn on 3 more lights, bringing the total to 6 lights on.
- The third button press would turn on 3 more lights, so now there would be 9 lights on.
- And so on . . .

Continuing in this way we can see that the total number of lights on would always be a multiple of 3. However, since 2020 is not a multiple of 3, this tells us that at least 1 button press must turn some lights off. Since we want to press the button the fewest number of times, that means we want the fewest number of button presses to turn lights off.

Now suppose the button was pressed 671 times, and each time 3 lights turned on. Then there would be $671 \times 3 = 2013$ lights on in total. Let's look at the remaining 7 lights that are still off. We can draw a diagram to show all the possible outcomes for the next button presses until all 7 lights are on.

Note that the order of the lights does not matter. We are interested in how many lights are on, not which particular lights are on. To simplify our diagram, at each step we have moved all of the lights that are on to the left.



Note that if a button press reverses the press that was just made, we did not include this in our diagram, as this will not give us the fewest number of button presses.

We can see in the diagram that the shortest sequence of steps to get all the lights on would be:

1. Turn 3 lights on.
2. Turn 1 light off and 2 lights on.
3. Turn 3 lights on.

This takes 3 steps, which means 3 button presses. If we add this to the 671 button presses to get to this point, that tells us there are $671 + 3 = 674$ button presses in total. We note that only 1 of these 674 button presses turns lights off. Since we know that at least 1 button press must turn some lights off, that tells us we cannot turn all the lights on using fewer button presses.

Therefore, 674 button presses are required to turn on all the lights.

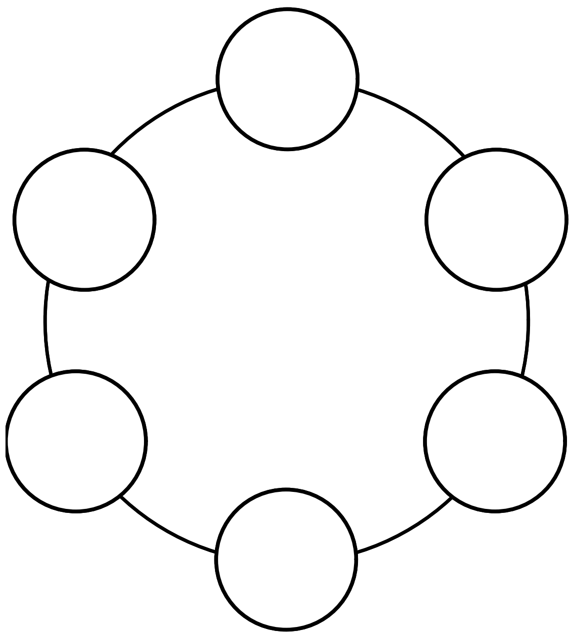
Problem of the Week

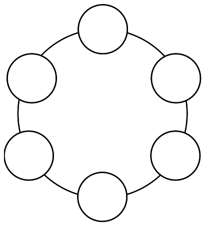
Problem D

A Circle of Numbers

The numbers 1, 6, 8, 13, 15, and 20 can be placed in the circle below, each exactly once, so that the sum of each pair of numbers adjacent in the circle is a multiple of seven.

In fact, there is more than one way to arrange the numbers in such a way in the circle. Determine all different arrangements. Note that we will consider two arrangements to be the same if one can be obtained from the other by a series of reflections and rotations.





Problem of the Week
Problem D and Solution
A Circle of Numbers

Problem

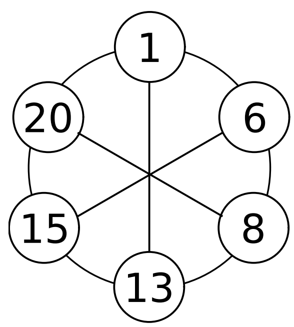
The numbers 1, 6, 8, 13, 15, and 20 can be placed in the circle above, each exactly once, so that the sum of each pair of numbers adjacent in the circle is a multiple of seven. In fact, there is more than one way to arrange the numbers in such a way in the circle. Determine all different arrangements. Note that we will consider two arrangements to be the same if one can be obtained from the other by a series of reflections and rotations.

Solution

We will start by writing down all the pairs of numbers that add to a multiple of 7.

Sum of 7	Sum of 14	Sum of 21	Sum of 28	Sum of 35
1,6	1,13	6,15	13,15	15,20
	6,8	8,13	8,20	
		1,20		

To show these connections visually, we can write the numbers in a circle and draw a line connecting numbers that add to a multiple of 7.

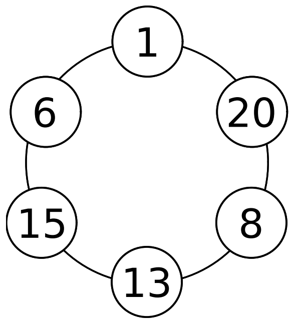


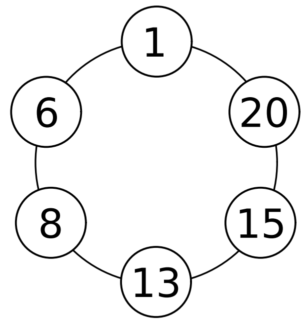
We will now determine all the different arrangements by looking at various cases. Note that in order for two arrangements to be different, at least some of the numbers need to be adjacent to different numbers.

Now, consider the possibilities for the numbers adjacent to 1. Since 6, 13, and 20 are the only numbers in our list that add with 1 to make a multiple of 7, there are three possible cases: 1 adjacent to 6 and 20, 1 adjacent to 6 and 13, and 1 adjacent to 13 and 20. We consider each case separately.

Case 1: 1 is adjacent to 6 and 20

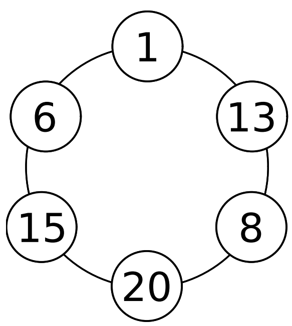
In this case, we can see from our table that 13 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.

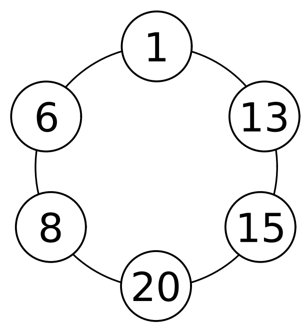




Case 2: 1 is adjacent to 6 and 13

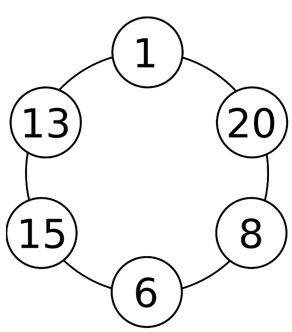
In this case, we can see from our table that 20 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.

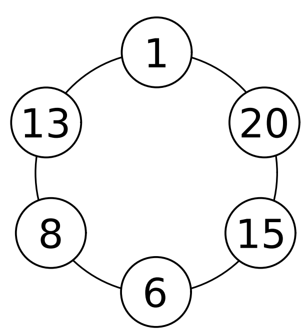




Case 3: 1 is adjacent to 13 and 20

In this case, we can see from our table that 6 must be adjacent to 15 and 8, since 1 is no longer available. The two different ways to write such a circle are shown below.





Therefore, we have found that there are 6 different arrangements. These are the arrangements shown in Cases 1, 2 and 3 above.