



The **CENTRE** for **EDUCATION**  
in **MATHEMATICS** and **COMPUTING**

**Problem of the Week**  
**Problems and Solutions**  
**2020-2021**

**Problem C (Grade 7/8)**

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**Themes**

**Number Sense (N)**

**Geometry (G)**

**Algebra (A)**

**Data Management (D)**

**Computational Thinking (C)**

(Click on a theme name above to jump to that section)

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\*The problems in this booklet are organized into themes.  
A problem often appears in multiple themes.

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# Number Sense (N)

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## Problem of the Week

### Problem C

#### This Product is a Mystery

The number  $A8$  is a two-digit number with tens digit  $A$  and units (ones) digit 8. Similarly,  $3B$  is a two-digit number with tens digit 3 and units digit  $B$ . When  $A8$  is multiplied by  $3B$ , the result is the four-digit number  $C730$ . That is,

$$\begin{array}{r} A8 \\ \times 3B \\ \hline C730 \end{array}$$



If  $A$ ,  $B$ , and  $C$  are each different digits from 0 to 9, determine the values of  $A$ ,  $B$ , and  $C$ .



## Problem of the Week

### Problem C and Solution

### This Product is a Mystery

#### Problem

The number  $A8$  is a two-digit number with tens digit  $A$  and units (ones) digit 8. Similarly,  $3B$  is a two-digit number with tens digit 3 and units digit  $B$ .

When  $A8$  is multiplied by  $3B$ , the result is the four-digit number  $C730$ . If  $A$ ,  $B$ , and  $C$  are each different digits from 0 to 9, determine the values of  $A$ ,  $B$ , and  $C$ .

#### Solution

In a multiplication question there are three parts: the *multiplier*, *multiplicand* and *product*. In our problem,  $A8$  is the multiplier,  $3B$  is the multiplicand, and  $C730$  is the product.

The units digit of the product  $C730$  is 0. The units digit of a product is equal to the units digit of the result obtained by multiplying the units digits of the multiplier and multiplicand.

So  $8 \times B$  must equal a number with units digit 0. The only choices for  $B$  are 0 and 5, since no other single digit multiplied by 8 produces a number ending in zero.

However, if  $B = 0$ , the units digit of the product is 0 and the remaining three digits of the product,  $C73$ , are produced by multiplying  $3 \times A8$ . But  $3 \times A8$  produces a number ending in 4, not 3 as required. Therefore  $B \neq 0$  and  $B$  must equal 5. So the multiplicand is 35.

Since  $A8$  is a two-digit number, the largest possible value for  $A$  is 9. Since  $98 \times 35 = 3430$ , the largest possible value of  $C$  is 3. Also, the product  $C730$  is a four-digit number so  $C \neq 0$ .

Therefore, the only possible values for  $C$  are 1, 2, and 3. We will examine each possibility for  $C$ .

If  $C = 1$ , then  $C730$  becomes 1730. We want  $A8 \times 35 = 1730$ . Alternatively, we want  $1730 \div 35 = A8$ . But  $1730 \div 35 \doteq 49.4$ , which is not a whole number. Therefore,  $C \neq 1$ .

If  $C = 2$ , then  $C730$  becomes 2730. We want  $A8 \times 35 = 2730$ . Alternatively, we want  $2730 \div 35 = A8$ . Since  $2730 \div 35 = 78$ , which is a whole number. Therefore,  $C = 2$  produces a valid value for  $A$ , namely  $A = 7$ .

If  $C = 3$ , then  $C730$  becomes 3730. We want  $A8 \times 35 = 3730$ . Alternatively, we want  $3730 \div 35 = A8$ . But  $3730 \div 35 \doteq 106.6$ , which is not a whole number. Therefore,  $C \neq 3$ .

We have examined every valid possibility for  $C$  and found only one solution. Therefore,  $A = 7$ ,  $B = 5$  and  $C = 2$  is the only valid solution. We can easily verify that  $78 \times 35 = 2730$ .



## Problem of the Week

### Problem C

### Homestyle Lemon Drink

A restaurant owner was experimenting with the taste of her homestyle lemon drink. She started with 60 litres of water. She removed 15 litres of the water and replaced it with 15 litres of pure lemon juice.

After thoroughly stirring the new mixture, she discovered that it was too “lemony”. So she removed 10 litres of the new mixture and replaced it with 10 litres of water. She thoroughly stirred the mixture and concluded that the new mixture was just right.

Determine the ratio of pure lemon juice to water in the final 60 litre mixture.





## Problem of the Week

### Problem C and Solution

### Homestyle Lemon Drink



#### Problem

A restaurant owner was experimenting with the taste of her homestyle lemon drink. She started with 60 litres of water. She removed 15 litres of the water and replaced it with 15 litres of pure lemon juice. After thoroughly stirring the new mixture, she discovered that it was too “lemony”. So she removed 10 litres of the new mixture and replaced it with 10 litres of water. She thoroughly stirred the mixture and concluded that the new mixture was just right. Determine the ratio of pure lemon juice to water in the final 60 litre mixture.

#### Solution

We need to determine the amount of pure lemon juice and the amount of water in the final mixture.

The owner starts with 60 litres of water and no lemon juice. After removing 15 litres of water and adding 15 litres of pure lemon juice, she has 15 litres of pure lemon juice and  $60 - 15 = 45$  litres of water. So  $\frac{15}{60} = \frac{1}{4}$  of the new mixture is pure lemon juice and  $\frac{45}{60} = \frac{3}{4}$  of the new mixture is water.

She then removes 10 litres of the new mixture,  $\frac{1}{4}$  of which is pure lemon juice and  $\frac{3}{4}$  of which is water. So the owner removes  $\frac{1}{4} \times 10 = \frac{10}{4} = \frac{5}{2}$  litres of pure lemon juice and  $\frac{3}{4} \times 10 = \frac{30}{4} = \frac{15}{2}$  litres of water.

Before adding another 10 litres of water she has  $15 - \frac{5}{2} = \frac{30}{2} - \frac{5}{2} = \frac{25}{2}$  litres of pure lemon juice and  $45 - \frac{15}{2} = \frac{90}{2} - \frac{15}{2} = \frac{75}{2}$  litres of water.

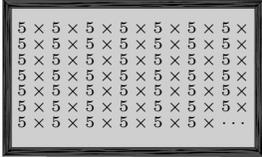
After adding the final 10 litres of water, she has  $10 + \frac{75}{2} = \frac{20}{2} + \frac{75}{2} = \frac{95}{2}$  litres of water.

The final ratio of pure lemon juice to water is

$$\frac{25}{2} : \frac{95}{2} = 25 : 95 = 5 : 19.$$

Therefore, the final ratio of pure lemon juice to water in the homestyle lemon drink is 5 : 19.





## Problem of the Week

### Problem C and Solution

#### More Power to You

#### Problem

In mathematics we like to write expressions concisely. For example, we will often write the expression  $5 \times 5 \times 5 \times 5$  as  $5^4$ . The lower number 5 is called the base, the raised 4 is called the exponent, and the whole expression  $5^4$  is called a power. So  $5^3$  means  $5 \times 5 \times 5$  and is equal to 125. What are the last three digits in the integer equal to  $5^{2020}$ ?

#### Solution

Let's start by examining the last three digits of various powers of 5.

$$5^1 = \quad \mathbf{005}$$

$$5^2 = \quad \mathbf{025}$$

$$5^3 = \quad \mathbf{125}$$

$$5^4 = \quad \mathbf{625}$$

$$5^5 = \quad \mathbf{3125}$$

$$5^6 = \quad \mathbf{15\,625}$$

$$5^7 = \quad \mathbf{78\,125}$$

$$5^8 = \quad \mathbf{390\,625}$$

Notice that there is a pattern for the last three digits after the first two powers of 5. For every odd integer exponent greater than 2, the last three digits are "125". For every even integer exponent greater than 2, the last three digits are "625". If the pattern continues, then  $5^9$  will end "125" since the exponent 9 is odd and  $5^{10}$  will end "625" since the exponent 10 is even. This is easily verified since  $5^9 = 1\,953\,125$  and  $5^{10} = 9\,765\,625$ .

We can easily justify why this pattern continues. If a power ends in "125", then the last 3 digits of the next power are the same as the last three digits of the product  $125 \times 5 = 625$ . That is, the last three digits of the next power are "625". If a power ends in "625", then the last 3 digits of the next power are the same as the last three digits of the product  $625 \times 5 = 3125$ . That is, the last three digits of the next power are "125".

For  $5^{2020}$ , the exponent 2020 is greater than 2 and is an even number.

Therefore, the last three digits of  $5^{2020}$  are 625.

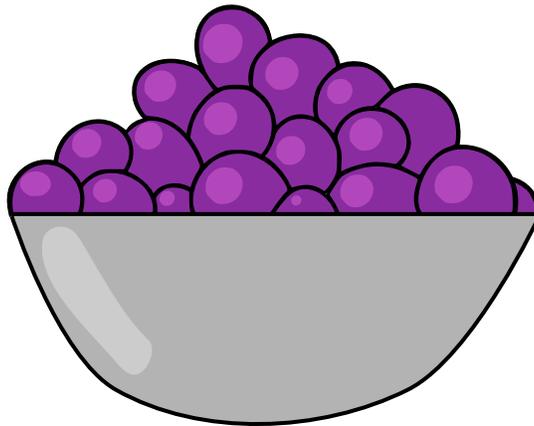


# Problem of the Week

## Problem C

### A Grape Problem

There are several bowls containing various amounts of grapes on a table. When 12 of the bowls each had 8 more grapes added to them, the mean (average) number of grapes per bowl increased by 6. How many bowls of grapes are on the table?





## Problem of the Week

### Problem C and Solution

#### A Grape Problem

#### Problem

There are several bowls containing various amounts of grapes on a table. When 12 of the bowls each had 8 more grapes added to them, the mean (average) number of grapes per bowl increased by 6. How many bowls of grapes are on the table?

#### Solution

##### Solution 1:

The mean number of grapes per bowl is equal to the total number of grapes divided by the number of bowls. So the mean number of grapes per bowl increases by 6 if the total number of grapes added divided by the number of bowls is equal to 6.

If 12 of the bowls each had 8 more grapes added to them, then the total number of grapes added is  $12 \times 8 = 96$ .

Thus, 96 divided by the number of bowls equals 6. Since  $96 \div 6 = 16$ , there are 16 bowls.

##### Solution 2:

This solution uses variables to represent the unknown values.

We are told that the mean number of grapes per bowl increased by 6. We can write this as follows.

$$\text{old mean} + 6 = \text{new mean}$$

The mean is equal to the total number of grapes divided by the number of bowls. Let  $b$  be the number of bowls. Let  $g$  be the total number of grapes that were *originally* in the bowls. If 12 of the bowls each had 8 more grapes added to them, then the total number of grapes added is  $12 \times 8 = 96$ . Now we can rewrite our equation above using variables.

$$\frac{g}{b} + 6 = \frac{g + 96}{b}$$

We can separate the fraction on the right side.

$$\frac{g}{b} + 6 = \frac{g}{b} + \frac{96}{b}$$

Now if we subtract  $\frac{g}{b}$  from both sides of the equation, we are left with an equation that has only one variable. We can then solve this equation.

$$6 = \frac{96}{b}$$

Multiply both sides by  $b$ :  $6b = 96$

Divide both sides by 6:  $b = \frac{96}{6}$

$$b = 16$$

Therefore, there are 16 bowls.



# Problem of the Week

## Problem C

### In Their Prime

A *prime number* is an integer greater than 1 with exactly two different positive factors, 1 and the number itself.

There are three children in a family. Each of their ages is a prime number. The sum of their ages is 41 and at least two of the children have ages that differ by 16.

Determine all possibilities for the ages of the children.





## Problem of the Week

### Problem C and Solution

#### In Their Prime

#### Problem

A *prime number* is an integer greater than 1 with exactly two different positive factors, 1 and the number itself. There are three children in a family. Each of their ages is a prime number. The sum of their ages is 41 and at least two of the children have ages that differ by 16. Determine all possibilities for the ages of the children.

#### Solution

We can start by listing all of the prime numbers less than 41. The possible prime ages are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and 37. We could actually eliminate some of the larger primes from this list since there are three different primes in the sum.

Now we will look for all prime pairs from this list that differ by 16. The pairs include 3 and 19, 7 and 23, and 13 and 29. We will look at each of these pairs and determine the third number so that the sum of the three ages is 41.

For the pair 3 and 19, the third age would be  $41 - 3 - 19 = 19$ , which is prime. The ages of the three children would be 3, 19, and 19. This is a possible solution.

For the pair 7 and 23, the third age would be  $41 - 7 - 23 = 11$ , which is prime. The ages of the three children would be 7, 11, and 23. This is another possible solution.

For the pair 13 and 29, the sum of these two ages is  $13 + 29 = 42$ . This sum is already over 41, so this is not a possible solution.

Therefore, there are two possibilities for the ages of the children. The children are either 7, 11 and 23 years old or 3, 19 and 19 years old.

For further thought: How would the problem change if the words “prime number” were removed from the problem and replaced with “positive integer”?



## Problem of the Week

### Problem C

#### How It Ends

The product of the positive integers 1 to 4 is

$$4 \times 3 \times 2 \times 1 = 24$$

and can be written in an abbreviated form as  $4!$ . We say “4 *factorial*”. So  $4! = 24$ .

The product of the positive integers 1 to 16 is

$$16 \times 15 \times 14 \times \cdots \times 3 \times 2 \times 1$$

and can be written in an abbreviated form as  $16!$ . We say “16 *factorial*”.

The  $\cdots$  represents the product of all the missing integers between 14 and 3.

In general, the product of the positive integers 1 to  $n$  is  $n!$ . Note that  $1! = 1$ .

Determine the tens digit and units (ones) digit of the sum

$$1! + 2! + 3! + \cdots + 2019! + 2020! + 2021!$$





## Problem of the Week

### Problem C and Solution

#### How It Ends

#### Problem

The product of the positive integers 1 to 4 is  $4 \times 3 \times 2 \times 1 = 24$  and can be written in an abbreviated form as  $4!$ . We say “4 factorial”. So  $4! = 24$ .

The product of the positive integers 1 to 16 is  $16 \times 15 \times 14 \times \cdots \times 3 \times 2 \times 1$  and can be written in an abbreviated form as  $16!$ . We say “16 factorial”.

The  $\cdots$  represents the product of all the missing integers between 14 and 3.

In general, the product of the positive integers 1 to  $n$  is  $n!$ . Note that  $1! = 1$ .

Determine the tens digit and units (ones) digit of the sum

$$1! + 2! + 3! + \cdots + 2019! + 2020! + 2021!$$

#### Solution

At first glance seems like there is a great deal of work to do. However, by examining several factorials, we will discover otherwise.

$$1! = 1$$

$$2! = 2 \times 1 = 2$$

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Now  $6! = 6 \times (5 \times 4 \times 3 \times 2 \times 1) = 6 \times 5! = 6(120) = 720$ ,

$$7! = 7 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 7 \times 6! = 7(720) = 5040,$$

$$8! = 8 \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 8 \times 7! = 8(5040) = 40\,320,$$

$$9! = 9 \times (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 9 \times 8! = 9(40\,320) = 362\,880, \text{ and}$$

$$10! = 10 \times (9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 10 \times 9! = 10(362\,880) = 3\,628\,800.$$

An interesting observation surfaces,  $9! = 9 \times 8!$ ,  $10! = 10 \times 9!$ ,  $11! = 11 \times 10!$ , and so on.

Furthermore, the last two digits of  $10!$  are 00. Every factorial above  $10!$  will also end with 00 since multiplying an integer that ends with 00 by another integer produces an integer product that ends in 00. So all factorials above  $10!$  will end with 00 and will not change the tens digit or the units digit in the required sum. We can determine the last two digits of the required sum by adding the last two digits of each of the factorials from  $1!$  to  $9!$ .

The sum of the last two digits of  $1!$  to  $9!$  will equal

$$1 + 2 + 6 + 24 + 20 + 20 + 40 + 20 + 80 = 213.$$

Therefore, for the sum  $1! + 2! + 3! + \cdots + 2019! + 2020! + 2021!$ , the tens digit will be 1 and the units (ones) digit will be 3. From what we have done, we do not know the hundreds digit.

**For Further Thought:** If we had only been interested in the units digit in the required sum, how many factorials would we need to calculate? If we wanted to know the last three digits, how many more factorials would be required?



## Problem of the Week

### Problem C

#### Painting the Way

Alexia, Benito, and Carmen won a team art competition at their local park. As part of their prize, they can paint the 5 km (5000 m) paved path through the park any way they like.

They decided Alexia will paint the first 70 m, then Benito will paint the next 15 m, then Carmen will paint the next 35 m. They will keep repeating this pattern until they reach the end of the 5 km path.

What percentage of the path will each person paint?





## Problem of the Week

### Problem C and Solution

#### Painting the Way

**Problem**

Alexia, Benito, and Carmen won a team art competition at their local park. As part of their prize, they can paint the 5 km (5000 m) paved path through the park any way they like. They decided Alexia will paint the first 70 m, then Benito will paint the next 15 m, then Carmen will paint the next 35 m. They will keep repeating this pattern until they reach the end of the 5 km path. What percentage of the path will each person paint?

**Solution**

After each person paints their first section, they will have covered  $70 + 15 + 35 = 120$  m in total. We will call this one cycle. The number of cycles in 5000 m is

$$\frac{5000}{120} = \frac{125}{3} = 41\frac{2}{3}.$$

So, there are 41 complete cycles and  $\frac{2}{3}$  of another cycle. The total distance covered in 41 cycles is  $41 \times 120 = 4920$  m. That means there are  $5000 - 4920 = 80$  m remaining. Alexia would paint the first 70 m, leaving the last 10 m for Benito to paint.

We can organize this information in a table to calculate the total distance each person will paint.

	Distance Painted by Each Person (m)		
	Alexia	Benito	Carmen
<b>First 41 cycles</b>	$41 \times 70 = 2870$	$41 \times 15 = 615$	$41 \times 35 = 1435$
<b>Last partial cycle</b>	70	10	0
<b>Total</b>	$2870 + 70 = 2940$	$615 + 10 = 625$	1435

We can now calculate the percentage of the path each person will paint.

$$\text{Alexia: } \frac{2940}{5000} \times 100\% = 58.8\%$$

$$\text{Benito: } \frac{625}{5000} \times 100\% = 12.5\%$$

$$\text{Carmen: } \frac{1435}{5000} \times 100\% = 28.7\%$$

Therefore, Alexia will paint 58.8% of the path, Benito will paint 12.5% of the path, and Carmen will paint 28.7% of the path.



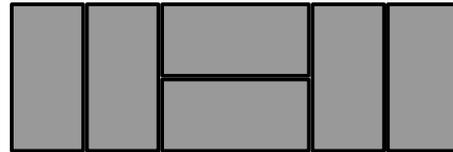
## Problem of the Week

### Problem C

#### A Path Using Math

A landscaper needs to fill a path measuring 2 feet by 6 feet with patio stones. The patio stones are each 1 foot by 2 feet, so the landscaper calculates that she will need 6 of them.

Before arranging the patio stones, the landscaper wants to look at all of her options. She cannot cut or overlap the stones, and they all must fit inside the path area without any gaps. Two possible arrangements of the stones are shown below. How many different arrangements are there in total?





## Problem of the Week

### Problem C and Solution

#### A Path Using Math



### Problem

A landscaper needs to fill a path measuring 2 feet by 6 feet with patio stones. The patio stones are each 1 foot by 2 feet, so the landscaper calculates that she will need 6 of them. Before arranging the patio stones, the landscaper wants to look at all of her options. She cannot cut or overlap the stones, and they all must fit inside the path area without any gaps. Two possible arrangements of the stones are shown. How many different arrangements are there in total?

### Solution

Let's consider the ways that the patio stones can be arranged. We will imagine we are looking at the path from the side, just like in the images shown in the question. First, notice that there must always be an even number of patio stones that have a horizontal orientation, because they must be placed in pairs.

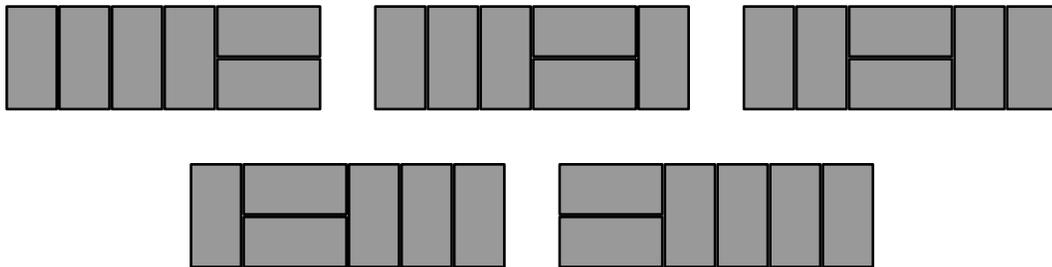
- All patio stones are vertical (and none are horizontal)

This can be done in only one way.



- Four patio stones are vertical and two are horizontal

There could be 0, 1, 2, 3 or 4 vertical stones to the right of the horizontal stones. So there are 5 ways that four patio stones are vertical and two are horizontal.



- Two patio stones are vertical and four are horizontal

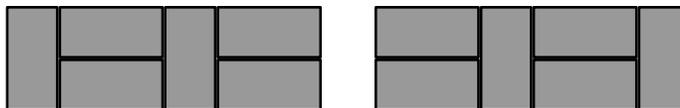
We need to consider sub cases:

- Case 1: There are no vertical stones between the horizontal stones.  
There could be 0, 1 or 2 vertical stones to the right of the horizontal stones. So there are 3 ways that two patio stones are vertical and four are horizontal when there are no vertical stones between the horizontal stones.





- Case 2: There is one vertical stone between the horizontal stones.  
There could be 0 or 1 vertical stones to the right of the rightmost horizontal stones.  
So there are 2 ways that two patio stones are vertical and four are horizontal when there is one vertical stone between the horizontal stones.

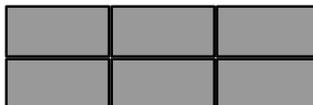


- Case 3: There are two vertical stones between the horizontal stones.  
There cannot be any vertical stones to the right of the rightmost horizontal stones, since all the vertical stones are between the horizontal stones. So there is 1 way that two patio stones are vertical and four are horizontal when there are two vertical stones between the horizontal stones.



- All patio stones are horizontal

This can be done in only one way.



Therefore, there are a total of  $1 + 5 + (3 + 2 + 1) + 1 = 13$  different arrangements of the patio stones.



## Problem of the Week

### Problem C

#### Greta's New Gig

Greta currently works 45 hours per week and earns a weekly salary of \$729. She will soon be starting a new job where her salary will be increased by 10% and her hours reduced by 10%.

How much more will she be earning per hour at her new job?





## Problem of the Week

### Problem C and Solution

#### Greta's New Gig

#### Problem

Greta currently works 45 hours per week and earns a weekly salary of \$729. She will soon be starting a new job where her salary will be increased by 10% and her hours reduced by 10%. How much more will she be earning per hour at her new job?

#### Solution

##### Solution 1

To calculate how much Greta earns per hour (i.e. her hourly rate of pay), divide her weekly salary by the number of hours worked.

Greta's old hourly rate of pay is  $\$729 \div 45 \text{ h} = \$16.20/\text{h}$ .

$$\begin{aligned}\text{New Weekly Salary} &= \text{Old Weekly Salary} + 10\% \text{ of Old Weekly Salary} \\ &= \$729 + 0.1 \times \$729 \\ &= \$729 + \$72.90 \\ &= \$801.90\end{aligned}$$

$$\begin{aligned}\text{New Number of Hours Worked} &= \text{Old Hours Worked} - 10\% \text{ of Old Hours Worked} \\ &= 45 \text{ h} - 0.1 \times 45 \text{ h} \\ &= 45 \text{ h} - 4.5 \text{ h} \\ &= 40.5 \text{ h}\end{aligned}$$

Greta's new hourly rate of pay is  $\$801.90 \div 40.5 \text{ h} = \$19.80/\text{h}$ .

The change in her hourly rate of pay is  $\$19.80/\text{h} - \$16.20/\text{h} = \$3.60/\text{h}$ .

Therefore, Greta will be earning \$3.60/h more at her new job.

##### Solution 2

In the second solution we will use a more concise calculation. Greta's new weekly salary is 10% more than her old weekly salary. So Greta will earn 110% of her old weekly salary. Greta's hours will be reduced by 10%, so her new hours will be 90% of her old hours. To calculate her change in hourly rate we can take her new hourly rate and subtract her old hourly rate.

$$\begin{aligned}\text{Change in Hourly Rate} &= \text{New Hourly Rate} - \text{Old Hourly Rate} \\ &= \text{New Salary} \div \text{New Hours Worked} - \text{Old Salary} \div \text{Old Hours Worked} \\ &= (\$729 \times 1.10) \div (45 \times 0.9) - \$729 \div 45 \\ &= \$801.90 \div 40.5 - \$729 \div 45 \\ &= \$19.80/\text{h} - \$16.20/\text{h} \\ &= \$3.60/\text{h}\end{aligned}$$

Therefore, Greta will be earning \$3.60/h more at her new job.



## Problem of the Week

### Problem C

#### A Large Product

Khushali selects three different numbers from the set  $\{-7, -5, -3, -1, 0, 2, 4, 6, 8\}$ . She then finds the product of the three chosen numbers. What is the largest product that Khushali can make?

$$\square \times \square \times \square$$



$$\square \times \square \times \square$$

## Problem of the Week

### Problem C and Solution

#### A Large Product

#### Problem

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#### Solution

Since  $4 \times 6 \times 8 = 192$ , then the greatest possible product is *at least* 192. In particular, the greatest possible product is positive.

For the product of three numbers to be positive, either all three numbers are positive or one number is positive and two numbers are negative. (If there were an odd number of negative factors, the product would be negative.)

If all three numbers are positive, the product is as large as possible when the three numbers are each as large as possible. In this case, the greatest possible product is  $4 \times 6 \times 8 = 192$ .

If one number is positive and two numbers are negative, their product is as large as possible if the positive number is as large as possible (8) and the product of the two negative numbers is as large as possible.

The product of the two negative numbers will be as large as possible when the negative numbers are each “as negative as possible” (that is, as far from 0 as possible). These numbers are thus  $-5$  and  $-7$  with product  $(-5) \times (-7) = 35$ . (We can check the other possible products of two negative numbers and see that none is as large.)

So the greatest possible product in this case is  $8 \times (-5) \times (-7) = 8 \times 35 = 280$ .

Combining the two cases, we see that the greatest possible product is 280.



## Problem of the Week

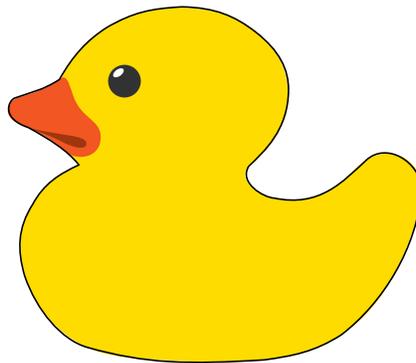
### Problem C

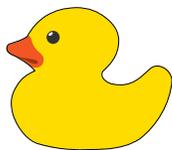
### Lucky Ducks

For a school fundraiser, Percy bought a big box of rubber ducks and wrote prize amounts, in dollars, on the bottom of each rubber duck. The prize amounts were \$5, \$10, \$20, \$50, and \$100. The number of ducks with each prize amount varied so that the total value for all the ducks with each prize amount was always \$500.

Percy then put all the rubber ducks in his school's swimming pool. At the fundraiser, participants used a long net to catch a duck from the pool. They won the amount written on the bottom of the rubber duck, and then they threw the duck back into the pool before the next person's turn. Since the ducks were returned to the pool after they were caught, the chances of winning any particular amount remained the same.

Amir paid \$15 to play the game once. If we assume he randomly selected a duck, what is the probability that Amir won more money than he paid to play the game?





## Problem of the Week

### Problem C and Solution

#### Lucky Ducks

#### Problem

For a school fundraiser, Percy bought a big box of rubber ducks and wrote prize amounts, in dollars, on the bottom of each rubber duck. The prize amounts were \$5, \$10, \$20, \$50, and \$100. The number of ducks with each prize amount varied so that the total value for all the ducks with each prize amount was always \$500.

Percy then put all the rubber ducks in his school's swimming pool. At the fundraiser, participants used a long net to catch a duck from the pool. They won the amount written on the bottom of the rubber duck, and then they threw the duck back into the pool before the next person's turn. Since the ducks were returned to the pool after they were caught, the chances of winning any particular amount remained the same.

Amir paid \$15 to play the game once. If we assume he randomly selected a duck, what is the probability that Amir won more money than he paid to play the game?

#### Solution

In order to determine the probability, we need to know the total number of ducks with each prize amount written on the bottom. These are calculated in the table below.

Prize Amount	Total Value of All Prizes with this Amount	Number of Ducks with this Prize Amount
\$5	\$500	$\$500 \div \$5 = 100$
\$10	\$500	$\$500 \div \$10 = 50$
\$20	\$500	$\$500 \div \$20 = 25$
\$50	\$500	$\$500 \div \$50 = 10$
\$100	\$500	$\$500 \div \$100 = 5$

The total number of ducks is  $100 + 50 + 25 + 10 + 5 = 190$ .

The total number of ducks containing a prize amount greater than \$15 is  $25 + 10 + 5 = 40$ .

Therefore, the probability that Amir won more than \$15 is:

$$\frac{\text{number of ducks containing a prize amount greater than \$15}}{\text{total number of ducks}} = \frac{40}{190} = \frac{4}{19}$$

That is, the probability that Amir won more than \$15 is  $\frac{4}{19}$ .

So Amir has approximately a 21% chance of winning more than he paid.



## Problem of the Week Problem C Step by Step

Debit and credit cards contain account numbers which consist of many digits. When purchasing items online, you are often asked to type in your account number. Because there are so many digits, it is easy to type the number incorrectly. The last digit of the number is a specially generated check digit which can be used to quickly verify the validity of the number. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid.

The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.

### Example 1:

Number: 135792

Reversal: 297531

$$A = 2 + 7 + 3 \\ = 12$$

$$2 \times 9 = 18$$

$$2 \times 5 = 10$$

$$2 \times 1 = 2$$

$$B = (1 + 8) + (1 + 0) + 2 \\ = 9 + 1 + 2 \\ = 12$$

$$C = 12 + 12 = 24$$

$C$  does not end in zero.  
The number is not valid.

### Example 2:

Number: 1357987

Reversal: 7897531

$$A = 7 + 9 + 5 + 1 \\ = 22$$

$$2 \times 8 = 16$$

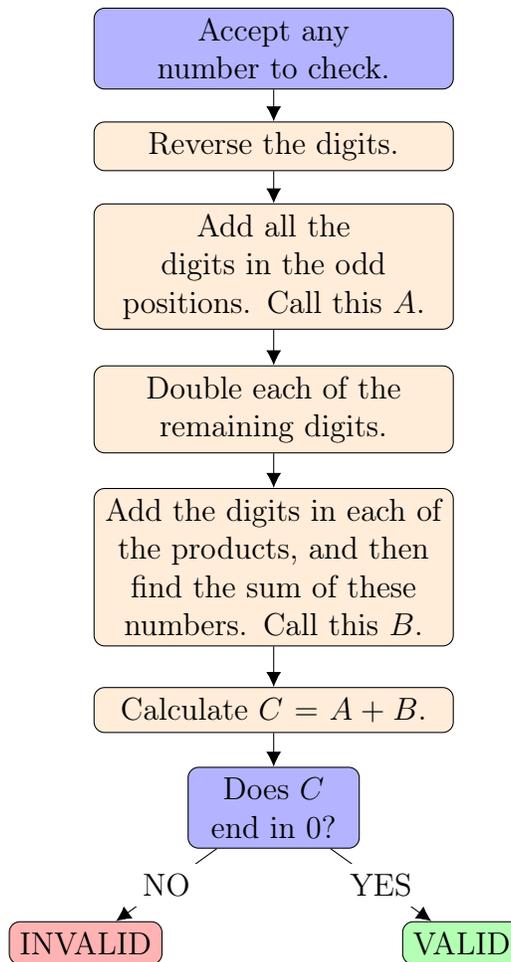
$$2 \times 7 = 14$$

$$2 \times 3 = 6$$

$$B = (1 + 6) + (1 + 4) + 6 \\ = 7 + 5 + 6 \\ = 18$$

$$C = 22 + 18 = 40$$

$C$  ends in zero.  
The number is valid.



The number 1953  $R$ 8 $T$ 9 467 is a valid card number when verified by the Luhn Algorithm.  $R$  and  $T$  are each single digits of the number such that  $R$  is less than  $T$ .

Determine all possible values of  $R$  and  $T$ .



## Problem of the Week

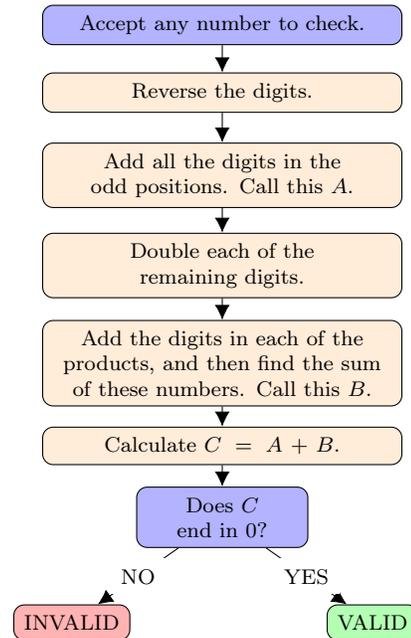
### Problem C and Solution

#### Step by Step

### Problem

Debit and credit cards contain account numbers which consist of many digits. When purchasing items online, you are often asked to type in your account number. Because there are so many digits, it is easy to type the number incorrectly. The last digit of the number is a specially generated check digit which can be used to quickly verify the validity of the number. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid.

The steps performed in the Luhn Algorithm are outlined in the flowchart to the right.



The number 1953 *RST*9 467 is a valid card number when verified by the Luhn Algorithm. *R* and *T* are each single digits of the number such that *R* is less than *T*. Determine all possible values of *R* and *T*.

### Solution

When the digits of the card number are reversed the resulting number is 764 9*T*8*R* 3591. The sum of the digits in the odd positions is  $A = 7 + 4 + T + R + 5 + 1 = 17 + R + T$ .

When the digits in the remaining positions are doubled, the following products are obtained:

$$6 \times 2 = 12; 9 \times 2 = 18; 8 \times 2 = 16; 3 \times 2 = 6; \text{ and } 9 \times 2 = 18$$

When the digit sums from each of the products are added, the sum is:

$$B = (1 + 2) + (1 + 8) + (1 + 6) + 6 + (1 + 8) = 3 + 9 + 7 + 6 + 9 = 34$$

Since  $C = A + B$ , we have  $C = 17 + R + T + 34 = 51 + R + T$ .

For the card to be valid, the units digit of  $C$  must be zero. The closest number greater than 51 ending in a zero is 60. It follows that  $R + T = 60 - 51 = 9$ . We want values of  $R$  and  $T$  that sum to 9 with  $R$  less than  $T$ . The only possible combinations are  $R = 0$  and  $T = 9$ ,  $R = 1$  and  $T = 8$ ,  $R = 2$  and  $T = 7$ ,  $R = 3$  and  $T = 6$ , and  $R = 4$  and  $T = 5$ .

The next closest number greater than 51 ending in a zero is 70. It follows that  $R + T = 70 - 51 = 19$ . But the largest value possible for  $T$  is 9 and the largest value for  $R$  would then be 8. It follows that the largest possible value for  $R + T = 8 + 9 = 17$ . This means that there are no values of  $R$  and  $T$  so that  $C$  could equal 70.

Therefore, there are only 5 possible pairs of digits for  $R$  and  $T$  so that 1953 *RST*9 467 is a valid card number. The possibilities are  $R = 0$  and  $T = 9$ ,  $R = 1$  and  $T = 8$ ,  $R = 2$  and  $T = 7$ ,  $R = 3$  and  $T = 6$ , and  $R = 4$  and  $T = 5$ .

The card numbers are 1953 0899 467, 1953 1889 467, 1953 2879 467, 1953 3869 467, and 1953 4859 467, which are indeed valid by the Luhn Algorithm.



## Problem of the Week

### Problem C

### CADET

We can take any word and rearrange all the letters to get another “word”. These new “words” may be nonsensical. For example, you can rearrange the letters in *MATH* to get *MTHA*. Geordie wants to rearrange all the letters in the word *CADET*. However, he uses the following rules:

- the letters *A* and *D* must be beside each other, and
- the letters *E* and *T* must be beside each other.

How many different arrangements of the word *CADET* can Geordie make if he follows these rules?



NOTE: *ADCET* and *ADTEC* are acceptable words, while *ADTCE* is not.



## Problem of the Week

### Problem C and Solution

### CADET

#### Problem

We can take any word and rearrange all the letters to get another “word”. These new “words” may be nonsensical. For example, you can rearrange the letters in *MATH* to get *MTHA*. Geordie wants to rearrange all the letters in the word *CADET*. However, he uses the following rules:

- the letters *A* and *D* must be beside each other, and
- the letters *E* and *T* must be beside each other.

How many different arrangements of the word *CADET* can Geordie make if he follows these rules?

#### Solution

We will look at a systematic way of counting the arrangements by first looking at a simpler example.

Let’s start with a three letter word. How many different ways can we arrange the letters of the word *SPY*?

If we list all the arrangements, we get 6 arrangements.

They are: *SPY, SYP, PYS, PSY, YPS, YSP*.

There is another way to count these 6 cases without listing them all out:

If we consider the first letter, there are 3 possibilities. For each of these possibilities, there are 2 remaining options for the second letter. Finally, once the first and second letters are set, there is only one possibility left for the last letter. To get the number of possible arrangements, we multiply  $\underline{3} \times \underline{2} \times \underline{1} = 6$ .

Let’s look at our problem now.

If we consider *A* and *D* as the single “letter” *AD*, and *E* and *T* as the single “letter” *ET*, we now have only the three “letters” *C, AD, and ET*.

As we saw above, there are  $\underline{3} \times \underline{2} \times \underline{1} = 6$  ways to arrange the three “letters”. These arrangements are:

*CADET, CETAD, ADCET, ADETC, ETCAD, ETADC*.

(We will refer to these as the original six.)

However, note that the question says *A* and *D* must be beside each other. This means they could appear as *AD* or *DA*. Similarly, *E* and *T* could appear as *ET* or *TE*.



Let's take a look at the first word, *CADET*.

We could switch *AD* to *DA*. This means *CADET* becomes *CDAET*, which is a valid arrangement.

We could also switch *ET* to *TE*. This means *CADET* becomes *CADTE*, which is a valid arrangement.

We could also switch both *AD* to *DA* and *ET* to *TE*. This means *CADET* becomes *CDATE*, which is a valid arrangement.

We can do these changes for each of the original six. We list these arrangements in the table below.

One of the Original Six	Switch <i>AD</i>	Switch <i>ET</i>	Switch both <i>AD</i> and <i>ET</i>
<i>CADET</i>	<i>CDAET</i>	<i>CADTE</i>	<i>CDATE</i>
<i>CETAD</i>	<i>CETDA</i>	<i>CTEAD</i>	<i>CTEDA</i>
<i>ADCET</i>	<i>DACET</i>	<i>ADCTE</i>	<i>DACTE</i>
<i>ADETC</i>	<i>DAETC</i>	<i>ADTEC</i>	<i>DATEC</i>
<i>ETCAD</i>	<i>ETCDA</i>	<i>TECAD</i>	<i>TECDA</i>
<i>ETADC</i>	<i>ETDAC</i>	<i>TEADC</i>	<i>TEDAC</i>

Therefore, there are 24 possible arrangements that Geordie can make when following the given rules.

NOTE: There is another way to count the number of arrangements. There are 6 ways to arrange the 3 "letters". There are 2 ways to arrange *AD* and there are 2 ways to arrange *ET*. To determine the total number of arrangements, we multiply  $6 \times 2 \times 2$ . This gives us 24 arrangements.



## Problem of the Week

### Problem C

#### Just Passing Through

A train is 1000 metres long. It is travelling at a constant speed, and approaches a tunnel that is 3000 metres long. From the time that the last car on the train has completely entered the tunnel until the time when the front of the train emerges from the other end, 30 seconds pass.

Determine the speed of the train, in kilometres per hour.





## Problem of the Week

### Problem C and Solution

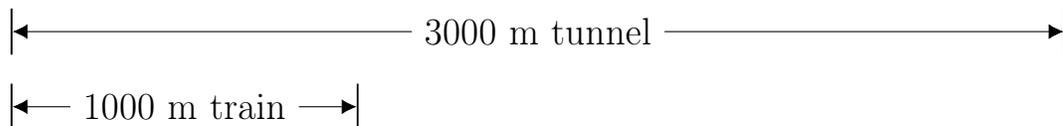
#### Just Passing Through

#### Problem

A train is 1000 metres long. It is travelling at a constant speed, and approaches a tunnel that is 3000 metres long. From the time that the last car on the train has completely entered the tunnel until the time when the front of the train emerges from the other end, 30 seconds pass. Determine the speed of the train, in kilometres per hour.

#### Solution

A diagram to represent the problem will make it easier to visualize.



At the time the entire train is just inside the tunnel, there is  $3000 - 1000 = 2000$  metres left to travel until the front of the train emerges from the other end. The train has to travel 2000 metres in 30 seconds. We can calculate the speed of the train by dividing the distance travelled by the time required to travel the distance.

The speed of the train is  $2000 \text{ m} \div 30 \text{ seconds} = \frac{200}{3} \text{ m/s}$ .

Now our task is to convert from m/s to km/h. We will do this in two steps: first convert metres to kilometres and then convert seconds to hours.

$$(1) \quad \frac{200 \text{ m}}{3 \text{ s}} = \frac{200 \text{ m}}{3 \text{ s}} \times \frac{1 \text{ km}}{1000 \text{ m}} = \frac{200 \text{ km}}{3000 \text{ s}} = \frac{1 \text{ km}}{15 \text{ s}}$$

$$(2) \quad \frac{1 \text{ km}}{15 \text{ s}} = \frac{1 \text{ km}}{15 \text{ s}} \times \frac{60 \text{ s}}{1 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ h}} = \frac{3600 \text{ km}}{15 \text{ h}} = \frac{240 \text{ km}}{1 \text{ h}}$$

The train is travelling at a speed of 240 km/h.



## Problem of the Week

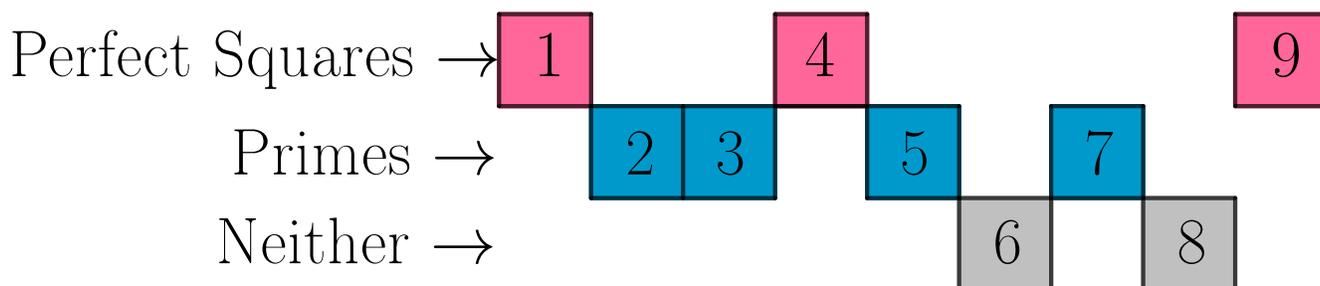
### Problem C

#### These Primes are Squares!

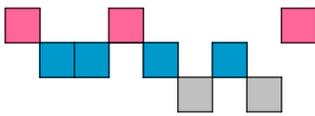
The number 7 has only two positive factors, 1 and itself. A positive integer greater than 1 whose only positive factors are 1 and itself is said to be *prime*.

A *perfect square* is an integer created by multiplying an integer by itself. The number 25 is a perfect square since it is  $5 \times 5$  or  $5^2$ .

Determine the smallest perfect square that has three different prime numbers as factors.



EXTENSION: Determine all perfect squares less than 10 000 that have three different prime numbers as factors.



## Problem of the Week

### Problem C and Solution

### These Primes are Squares!

#### Problem

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A *perfect square* is an integer created by multiplying an integer by itself. The number 25 is a perfect square since it is  $5 \times 5$  or  $5^2$ .

Determine the smallest perfect square that has three different prime numbers as factors.

EXTENSION: Determine all perfect squares less than 10 000 that have three different prime numbers as factors.

#### Solution

The problem itself is not very difficult once you determine what it is asking. Let's begin by examining some perfect squares.

The numbers 4 and 9 are both perfect squares that have only one prime number as a factor,  $4 = 2^2$  and  $9 = 3^2$ . The number 36 is a perfect square since  $36 = 6^2$ . However, the number  $6 = 2 \times 3$  so  $36 = (2 \times 3)^2 = 2 \times 3 \times 2 \times 3 = 2^2 \times 3^2$ . So 36 is the product of the squares of two different prime numbers.

Notice that when we write a perfect square as a product of prime factors, each prime factor appears an even number of times in the product. This is because the perfect square is created by multiplying an integer by itself, so all of the prime factors of the integer appear twice.

To create the smallest perfect square with three different prime factors, we should choose the three smallest prime numbers, namely 2, 3, and 5. If we include any primes larger than these, then their product will be larger and so the perfect square will be larger. To make our number a perfect square, we should have each prime factor appear twice. This gives us  $2^2 \times 3^2 \times 5^2 = 4 \times 9 \times 25 = 900$ . It should be noted that  $900 = 30^2 = (2 \times 3 \times 5)^2$ .

Therefore, the smallest perfect square with three different prime numbers as factors is 900.

#### EXTENSION ANSWER:

Since perfect squares have an even number of each prime factor in their factorization, we can find all the perfect squares less than 10 000 with three different prime factors as numbers by systematically trying different combinations of prime numbers. The table below lists all the perfect squares less than 10 000 with three different prime factors.

$2^2 \times 3^2 \times 5^2 = 900$	$2^2 \times 2^2 \times 3^2 \times 5^2 = 3600$	$2^2 \times 3^2 \times 3^2 \times 5^2 = 8100$
$2^2 \times 3^2 \times 7^2 = 1764$	$2^2 \times 2^2 \times 3^2 \times 7^2 = 7056$	
$2^2 \times 3^2 \times 11^2 = 4356$		
$2^2 \times 3^2 \times 13^2 = 6084$		
$2^2 \times 5^2 \times 7^2 = 4900$		

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# Geometry (G)

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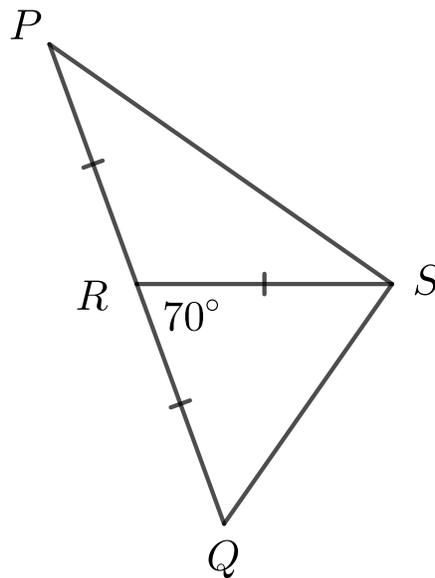
# Problem of the Week

## Problem C

### Angled

In  $\triangle PQS$  below,  $R$  lies on  $PQ$  such that  $PR = RQ = RS$  and  $\angle QRS = 70^\circ$ .

Determine the measure of  $\angle PSQ$ .

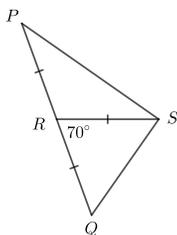




## Problem of the Week

### Problem C and Solution

### Angled



#### Problem

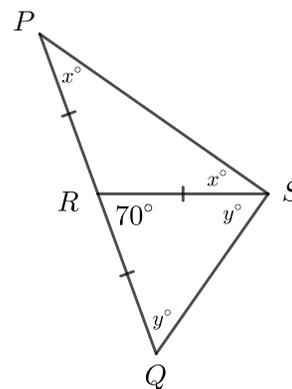
In  $\triangle PQS$  above,  $R$  lies on  $PQ$  such that  $PR = RQ = RS$  and  $\angle QRS = 70^\circ$ . Determine the measure of  $\angle PSQ$ .

#### Solution

##### Solution 1

In  $\triangle PRS$ , since  $PR = RS$ ,  $\triangle PRS$  is isosceles and  $\angle RPS = \angle RSP = x^\circ$ .

Similarly, in  $\triangle QRS$ , since  $RQ = RS$ ,  $\triangle QRS$  is isosceles and  $\angle RQS = \angle RSQ = y^\circ$ .



Since  $PRQ$  is a straight line,  $\angle PRS + \angle QRS = 180^\circ$ . Since  $\angle QRS = 70^\circ$ , we have  $\angle PRS = 110^\circ$ .

The angles in a triangle sum to  $180^\circ$ , so in  $\triangle PRS$

$$\begin{aligned}\angle RPS + \angle RSP + \angle PRS &= 180^\circ \\ x^\circ + x^\circ + 110^\circ &= 180^\circ \\ 2x &= 70 \\ x &= 35\end{aligned}$$

The angles in a triangle sum to  $180^\circ$ , so in  $\triangle QRS$

$$\begin{aligned}\angle RQS + \angle RSQ + \angle QRS &= 180^\circ \\ y^\circ + y^\circ + 70^\circ &= 180^\circ \\ 2y &= 110 \\ y &= 55\end{aligned}$$

Then  $\angle PSQ = \angle RSP + \angle RSQ = x^\circ + y^\circ = 35^\circ + 55^\circ = 90^\circ$ .

Therefore, the measure of  $\angle PSQ$  is  $90^\circ$ .

See Solution 2 for a more general approach to the solution of this problem.

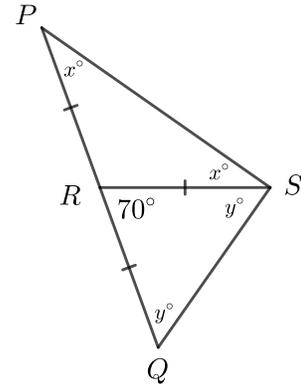


It turns out that it is not necessary to determine the values of  $x$  and  $y$  to solve this problem.

### Solution 2

In  $\triangle PRS$ , since  $PR = RS$ ,  $\triangle PRS$  is isosceles and  $\angle RPS = \angle RSP = x^\circ$ .

Similarly, in  $\triangle QRS$ , since  $RQ = RS$ ,  $\triangle QRS$  is isosceles and  $\angle RQS = \angle RSQ = y^\circ$ .



The angles in a triangle sum to  $180^\circ$ , so in  $\triangle PQS$

$$\angle QPS + \angle PSQ + \angle PQS = 180^\circ$$

$$x^\circ + (x^\circ + y^\circ) + y^\circ = 180^\circ$$

$$(x^\circ + y^\circ) + (x^\circ + y^\circ) = 180^\circ$$

$$2(x^\circ + y^\circ) = 180^\circ$$

$$x^\circ + y^\circ = 90^\circ$$

But  $\angle PSQ = \angle RSP + \angle RSQ = x^\circ + y^\circ = 90^\circ$ .

Therefore, the measure of  $\angle PSQ$  is  $90^\circ$ .



## Problem of the Week

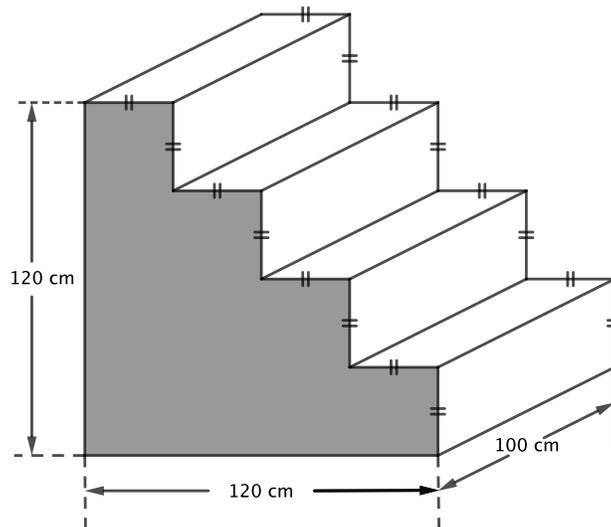
### Problem C

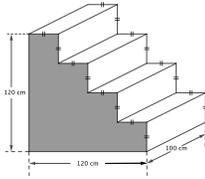
### New Steps

The POTW clubhouse needs to replace their front steps. A suggestion for the steps is shown below. The steps are of equal depth and equal height. The steps are 120 cm high, 120 cm from front to back and 100 cm wide.

The stairs are to be painted gold and the two sides are to be painted black. One of the two sides that is to be painted black is shaded in the diagram. The back and bottom of the structure will not be seen and will not be painted.

Determine the total area to be painted gold and the total area to be painted black.





## Problem of the Week

### Problem C and Solution

#### New Steps

#### Problem

The POTW clubhouse needs to replace their front steps. A suggestion for the steps is shown below. The steps are of equal depth and equal height. The steps are 120 cm high, 120 cm from front to back and 100 cm wide. The stairs are to be painted gold and the two sides are to be painted black. One of the two sides that is to be painted black is shaded in the diagram. The back and bottom of the structure will not be seen and will not be painted.

Determine the total area to be painted gold and the total area to be painted black.

#### Solution

In order to solve both parts of the problem, the depth and height of each individual step must be calculated. The entire structure is 120 cm from front to back and four steps cover the entire depth. Therefore, each step is  $120 \text{ cm} \div 4 = 30 \text{ cm}$  wide and high.

The area to paint gold is made up of 8 identical rectangles, each 30 cm wide and 100 cm long. Using the formula  $\text{area} = \text{length} \times \text{width}$ , the area of one rectangle is  $30 \times 100 = 3000 \text{ cm}^2$ .

The area of all surfaces to be painted gold is  $8 \times 3000 = 24000 \text{ cm}^2$ .

There are many different ways to find the area of the sides of the stairs. One solution would be to break the figure into four equal width rectangles, one four steps high, one three steps high, one two steps high and the final rectangle one step high.

The area of one side is

$$30 \times (4 \times 30) + 30 \times (3 \times 30) + 30 \times (2 \times 30) + 30 \times (1 \times 30) \\ = 30 \times 120 + 30 \times 90 + 30 \times 60 + 30 \times 30 = 3600 + 2700 + 1800 + 900 \\ = 9000 \text{ cm}^2.$$

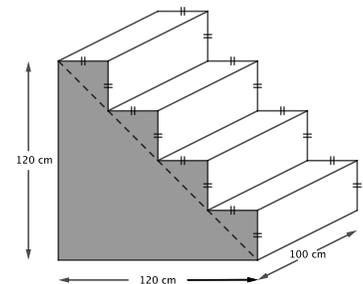
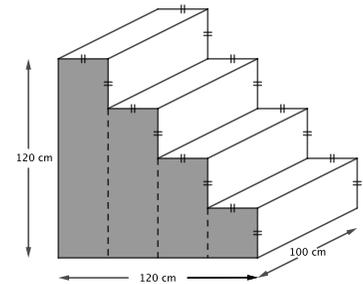
The total area of the two sides to be painted black is  $2 \times 9000 = 18000 \text{ cm}^2$

A second method to calculate the area of one side involves breaking the figure into triangles. Draw a diagonal from the top left corner to the bottom right corner. This diagonal line would hit the bottom corner of each step as shown in the diagram. The larger triangle has a base and height of 120 cm. Each of the four smaller triangles has a base and a height of 30 cm, the width and height of each step.

Using the formula  $\text{area} = \text{base} \times \text{height} \div 2$ , the area of one side is  $120 \times 120 \div 2 + 4 \times (30 \times 30 \div 2) = 7200 + 1800 = 9000 \text{ cm}^2$ .

The total area of the two sides to be painted black is  $2 \times 9000 = 18000 \text{ cm}^2$ .

**Therefore, the total area to paint gold is  $24000 \text{ cm}^2$  and the total area to paint black is  $18000 \text{ cm}^2$ .**



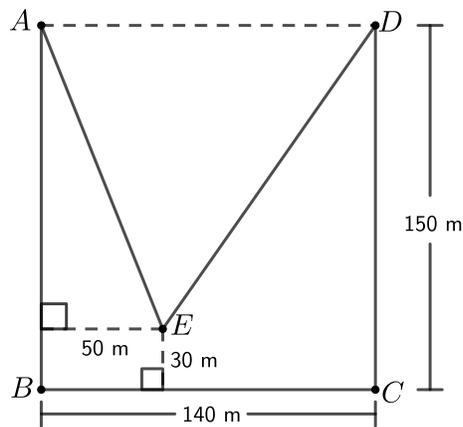


## Problem of the Week

### Problem C

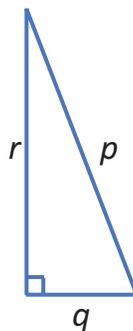
### Around the Farm

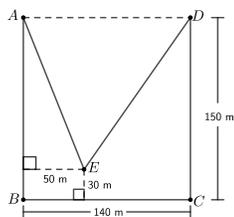
Rahul has a farm he wishes to fence. The farm is the pentagon  $ABCDE$ , shown below. He knows that  $ABCD$  is a 140 m by 150 m rectangle, as shown below. He also knows that  $E$  is 50 m from the side  $AB$  and 30 m from the side  $BC$ . Determine the length of  $AE$ , the length of  $DE$ , and the perimeter of pentagon  $ABCDE$ .



NOTE: The *Pythagorean Theorem* states, “In a right-angled triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides”.

In the following right triangle,  $p^2 = r^2 + q^2$ .





## Problem of the Week

### Problem C and Solution

#### Around the Farm

#### Problem

Rahul has a farm he wishes to fence. The farm is the pentagon  $ABCDE$ , shown above. He knows that  $ABCD$  is a 140 m by 150 m rectangle, as shown below. He also knows that  $E$  is 50 m from the side  $AB$  and 30 m from the side  $BC$ .

Determine the length of  $AE$ , the length of  $DE$ , and the perimeter of pentagon  $ABCDE$ .

#### Solution

Let  $F$  be the point on  $AB$  with  $EF = 50$  m.

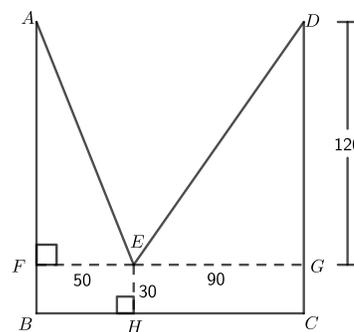
Let  $H$  be the point on  $BC$  with  $EH = 30$  m.

Extend  $EF$  to  $G$  on  $CD$ .

Since  $ABCD$  is a rectangle and  $FG$  is perpendicular to  $AB$ , then  $FG$  is perpendicular to  $CD$  and  $FGCB$  is a rectangle.

Therefore,  $FB = EH = GC = 30$  m.

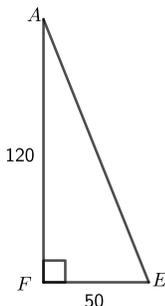
Also,  $DG = AF = AB - FB = 150 - 30 = 120$  m.



Since  $\triangle AFE$  and  $\triangle DGE$  are right-angled triangles, we can use the Pythagorean Theorem to determine the lengths of  $AE$  and  $DE$ .

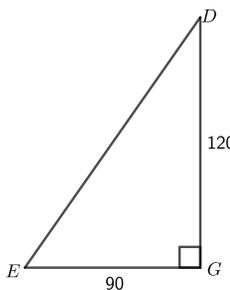
In  $\triangle AFE$ ,

$$\begin{aligned} AE^2 &= AF^2 + FE^2 \\ &= 120^2 + 50^2 \\ &= 14\,400 + 2\,500 \\ &= 16\,900 \\ AE &= 130, \text{ since } AE > 0 \end{aligned}$$



In  $\triangle DGE$ ,

$$\begin{aligned} DE^2 &= DG^2 + EG^2 \\ &= 120^2 + 90^2 \\ &= 14\,400 + 8\,100 \\ &= 22\,500 \\ DE &= 150, \text{ since } DE > 0 \end{aligned}$$



Therefore,  $AE = 130$  m and  $DE = 150$  m.

Also, the perimeter of pentagon  $ABCDE$  is equal to

$$AB + BC + CD + DE + AE = 150 + 140 + 150 + 150 + 130 = 720 \text{ m.}$$



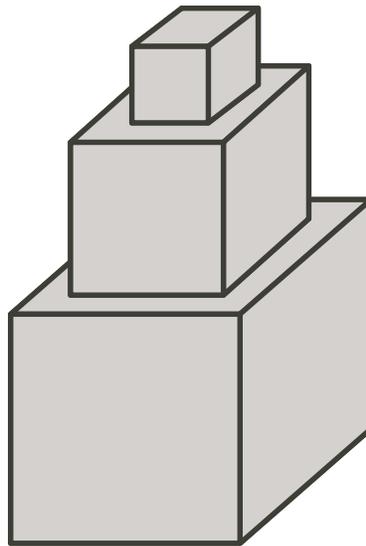
## Problem of the Week

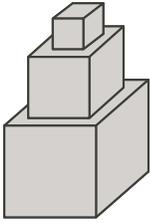
### Problem C

#### 'Cube'ism

As an artist, Ernest is doing a solo show titled CUBES. One of his pieces is shown below. Each layer of the piece is a cube. The bottom cube has a side length of 3 m, the middle cube has a side length of 2 m, and the top cube has a side length of 1 m. The top two layers are each centred on the layer below.

Ernest wishes to paint the piece. Since the piece will be suspended in the air, the bottom will also be painted. Determine the total surface area of the piece, including the bottom.





## Problem of the Week

### Problem C and Solution

#### 'Cube'ism

#### Problem

As an artist, Ernest is doing a solo show titled CUBES. One of his pieces is shown below. Each layer of the piece is a cube. The bottom cube has a side length of 3 m, the middle cube has a side length of 2 m, and the top cube has a side length of 1 m. The top two layers are each centred on the layer below. Ernest wishes to paint the piece. Since the piece will be suspended in the air, the bottom will also be painted. Determine the total surface area of the piece, including the bottom.

#### Solution

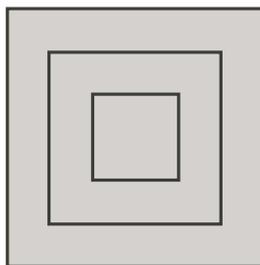
To determine the areas we will primarily use the formula for the area of a rectangle

$$\text{Area} = \text{length} \times \text{width}$$

Each cube has 4 exposed square sides, so the total area of all the sides is

$$\begin{aligned} 4 \times (1 \times 1) + 4 \times (2 \times 2) + 4 \times (3 \times 3) &= 4 \times (1) + 4 \times (4) + 4 \times (9) \\ &= 4 + 16 + 36 \\ &= 56 \text{ m}^2 \end{aligned}$$

To determine the area of the exposed top of each of the cubes, look down on the tower and see a image like the one below.



This exposed area is exactly the same as the side area of one face of the largest cube. Therefore, the top exposed area is  $3 \times 3 = 9 \text{ m}^2$ . The top area and the bottom area are the same. Therefore, the bottom area is  $9 \text{ m}^2$ .

**Therefore, the total surface area is  $56 + 9 + 9 = 74 \text{ m}^2$ .**

**Extension:** Three cubes with side lengths  $x$ ,  $y$  and  $z$  are stacked on top of each other in a similar manner to the original problem such that  $0 < x < y < z$ . Show that the total surface area of the stack, including the bottom, is  $6z^2 + 4y^2 + 4x^2$ .



## Problem of the Week

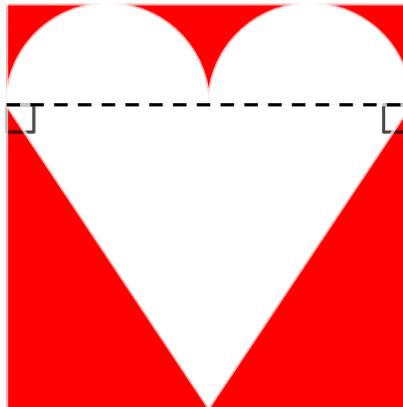
### Problem C

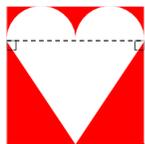
#### A Homemade Expression of Love

Maggie wants to make a special card for Valentine's Day. She starts with a square piece of red paper with a side length of 12 cm. She then pastes two white semi-circles, each with radius 3 cm, and a white triangle onto the square sheet of red paper, as shown below. (The dashed line and the right angle symbols will not actually be on the finished card.).

She is going to write her valentine a message in red ink on the white region of the card.

Determine the total amount of area available in the white region for Maggie's special valentine greeting.





## Problem of the Week

### Problem C and Solution

#### A Homemade Expression of Love

#### Problem

Maggie wants to make a special card for Valentine's Day. She starts with a square piece of red paper with a side length of 12 cm. She then pastes two white semi-circles, each with radius 3 cm, and a white triangle onto the square sheet of red paper, as shown below. (The dashed line and the right angle symbols will not actually be on the finished card.). She is going to write her valentine a message in red ink on the white region of the card. Determine the total amount of area available in the white region for Maggie's special valentine greeting.

#### Solution

The given information is shown on the diagram to the right.

The total area for writing the message is the area of the two semi-circles plus the area of the white triangle.

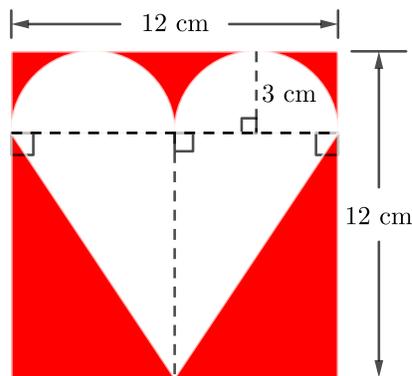
Since there are two semi-circles of radius 3 cm, the total area of the two semi-circles is equal to the area of a full circle of radius 3 cm. Therefore, the area of the two semi-circles is

$$\pi r^2 = \pi(3)^2 = 9\pi \text{ cm}^2.$$

The height of the triangle is the length of the square minus the radius of a semi-circle. Therefore the height of the triangle is  $12 - 3 = 9$  cm. The base of the triangle is 12 cm, the width of the square. The area of the triangle is

$$\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(12)(9) = 54 \text{ cm}^2.$$

The total available area for Maggie's message is  $(9\pi + 54) \text{ cm}^2$ . This area is approximately  $82.3 \text{ cm}^2$ . Happy Valentine's Day.



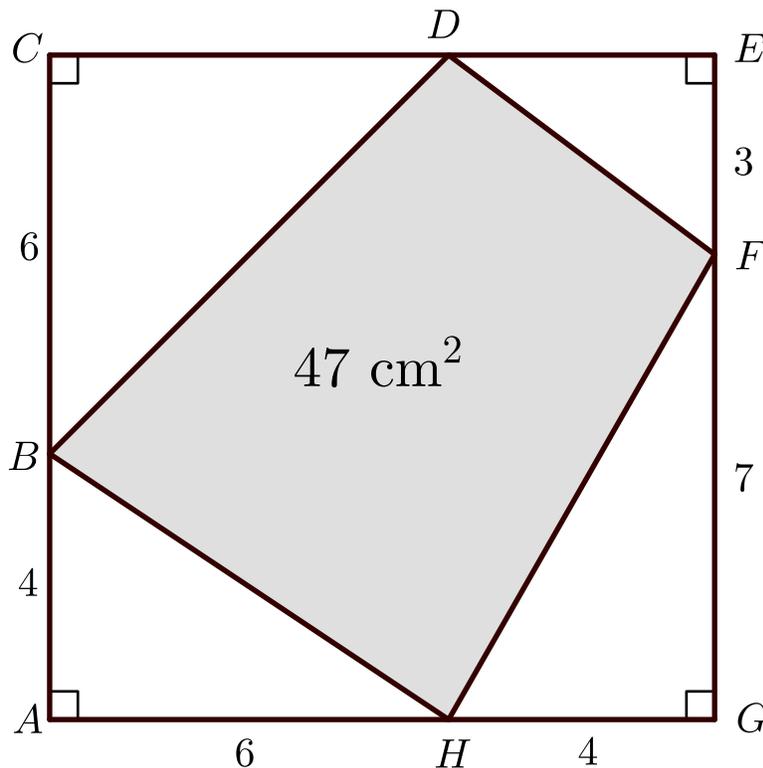


**Problem of the Week**  
**Problem C**  
**Locate the Fourth Vertex**

Quadrilateral  $BDFH$  is constructed so that each vertex is on a different side of square  $ACEG$ . Vertex  $B$  is on side  $AC$  so that  $AB = 4$  cm and  $BC = 6$  cm. Vertex  $F$  is on  $EG$  so that  $EF = 3$  cm and  $FG = 7$  cm. Vertex  $H$  is on  $GA$  so that  $GH = 4$  cm and  $HA = 6$  cm. The area of quadrilateral  $BDFH$  is  $47$  cm<sup>2</sup>.

The fourth vertex of quadrilateral  $BDFH$ , labelled  $D$ , is located on side  $CE$  so that the lengths of  $CD$  and  $DE$  are both positive integers.

Determine the lengths of  $CD$  and  $DE$ .





## Problem of the Week Problem C and Solution Locate the Fourth Vertex

### Problem

Quadrilateral  $BDFH$  is constructed so that each vertex is on a different side of square  $ACEG$ . Vertex  $B$  is on side  $AC$  so that  $AB = 4$  cm and  $BC = 6$  cm. Vertex  $F$  is on  $EG$  so that  $EF = 3$  cm and  $FG = 7$  cm. Vertex  $H$  is on  $GA$  so that  $GH = 4$  cm and  $HA = 6$  cm. The area of quadrilateral  $BDFH$  is  $47$   $\text{cm}^2$ .

The fourth vertex of quadrilateral  $BDFH$ , labelled  $D$ , is located on side  $CE$  so that the lengths of  $CD$  and  $DE$  are both positive integers.

Determine the lengths of  $CD$  and  $DE$ .

### Solution

#### Solution 1

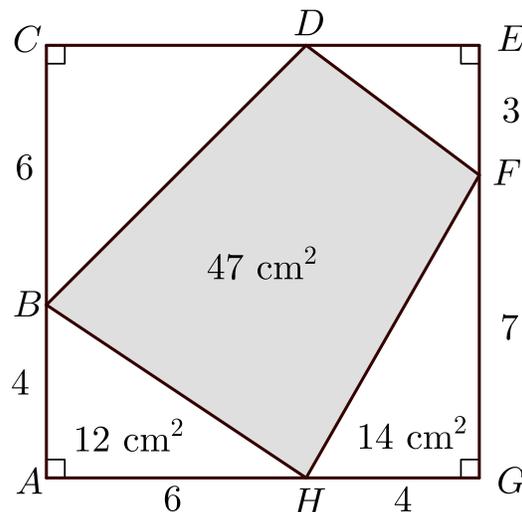
Since  $ACEG$  is a square and the length of  $AG = AH + HG = 6 + 4 = 10$  cm, then the length of each side of the square is 10 cm and the area is  $10 \times 10 = 100$   $\text{cm}^2$ .

We can determine the area of triangles  $BAH$  and  $FGH$  using the formula  $\text{area} = \frac{\text{base} \times \text{height}}{2}$ .

In  $\triangle BAH$ , since  $BA$  is perpendicular to  $AH$ , we can use  $BA$  as the height and  $AH$  as the base. The area of  $\triangle BAH$  is  $\frac{6 \times 4}{2} = 12$   $\text{cm}^2$ .

In  $\triangle FGH$ , since  $FG$  is perpendicular to  $GH$ , we can use  $FG$  as the height and  $GH$  as the base. The area of  $\triangle FGH$  is  $\frac{4 \times 7}{2} = 14$   $\text{cm}^2$ .

The area of  $\triangle BCD$  plus the area of  $\triangle FED$  must be the total area minus the three known areas. That is,  
 $\text{Area } \triangle BCD + \text{Area } \triangle FED = 100 - 12 - 47 - 14 = 27$   $\text{cm}^2$ .



$CD$  and  $DE$  are both positive integers and  $CD + DE = 10$ . We will systematically check all possible values for  $CD$  and  $DE$  to determine the values which produce the correct area.

$CD$	$DE$	Area $\triangle BCD$	Area $\triangle FED$	Area $\triangle BCD + \text{Area } \triangle FED$
1	9	$1 \times 6 \div 2 = 3$	$9 \times 3 \div 2 = 13.5$	$3 + 13.5 = 16.5 \neq 27$
2	8	$2 \times 6 \div 2 = 6$	$8 \times 3 \div 2 = 12$	$6 + 12 = 18 \neq 27$
3	7	$3 \times 6 \div 2 = 9$	$7 \times 3 \div 2 = 10.5$	$9 + 10.5 = 19.5 \neq 27$
4	6	$4 \times 6 \div 2 = 12$	$6 \times 3 \div 2 = 9$	$12 + 9 = 21 \neq 27$
5	5	$5 \times 6 \div 2 = 15$	$5 \times 3 \div 2 = 7.5$	$15 + 7.5 = 22.5 \neq 27$
6	4	$6 \times 6 \div 2 = 18$	$4 \times 3 \div 2 = 6$	$18 + 6 = 24 \neq 27$
7	3	$7 \times 6 \div 2 = 21$	$3 \times 3 \div 2 = 4.5$	$21 + 4.5 = 25.5 \neq 27$
8	2	$8 \times 6 \div 2 = 24$	$2 \times 3 \div 2 = 3$	$24 + 3 = 27$
9	1	$9 \times 6 \div 2 = 27$	$1 \times 3 \div 2 = 1.5$	$27 + 1.5 = 28.5 \neq 27$

Therefore, when  $CD = 8$  cm and  $DE = 2$  cm, the area of quadrilateral  $BDFH$  is  $47$   $\text{cm}^2$ .

The second solution is more algebraic and will produce a solution for any lengths of  $CD$  and  $DE$  between 0 and 10 cm.



**Solution 2**

This solution begins the same as Solution 1. Algebra is introduced to complete the solution.

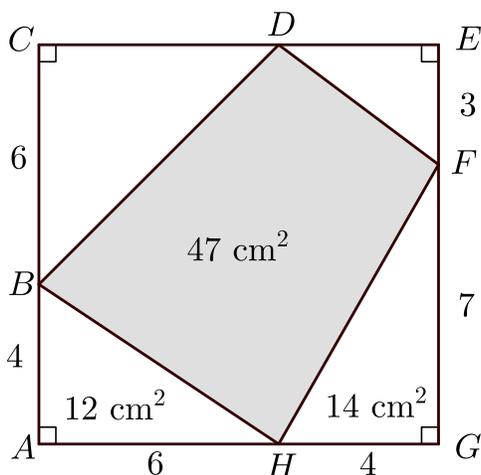
Since  $ACEG$  is a square and the length of  $AG = AH + HG = 6 + 4 = 10$  cm, then the length of each side of the square is 10 cm and the area is  $10 \times 10 = 100$  cm<sup>2</sup>.

We can determine the area of the triangles  $BAH$  and  $FGH$  using the formula  $\frac{\text{base} \times \text{height}}{2}$ .

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In  $\triangle FGH$ , since  $FG$  is perpendicular to  $GH$ , we can use  $FG$  as the height and  $GH$  as the base. The area of  $\triangle FGH$  is  $\frac{4 \times 7}{2} = 14$  cm<sup>2</sup>.

The area of  $\triangle BCD$  plus the area of  $\triangle FED$  must be the total area minus the three known areas. That is, Area  $\triangle BCD$  + Area  $\triangle FED = 100 - 12 - 47 - 14 = 27$  cm<sup>2</sup>.



Let the length of  $CD$  be  $n$  cm. Then the length of  $DE$  is  $(10 - n)$  cm.

The area of  $\triangle BCD$  is  $\frac{BC \times CD}{2} = \frac{6 \times n}{2} = 3n$ .

The area of  $\triangle FED$  is  $\frac{FE \times DE}{2} = \frac{3 \times (10 - n)}{2} = \frac{10 - n + 10 - n + 10 - n}{2} = \frac{30 - 3n}{2}$ .

Therefore,

$$\text{Area } \triangle BCD + \text{Area } \triangle FED = 27$$

$$3n + \frac{30 - 3n}{2} = 27$$

Multiplying both sides by 2:

$$6n + 30 - 3n = 54$$

$$3n + 30 = 54$$

$$3n = 24$$

$$n = 8$$

Therefore, the length of  $CD$  is 8 cm and the length of  $DE$  is 2 cm.

The algebra presented in Solution 2 may not be familiar to all students at this level.



## Problem of the Week

### Problem C

#### Just Passing Through

A train is 1000 metres long. It is travelling at a constant speed, and approaches a tunnel that is 3000 metres long. From the time that the last car on the train has completely entered the tunnel until the time when the front of the train emerges from the other end, 30 seconds pass.

Determine the speed of the train, in kilometres per hour.





## Problem of the Week

### Problem C and Solution

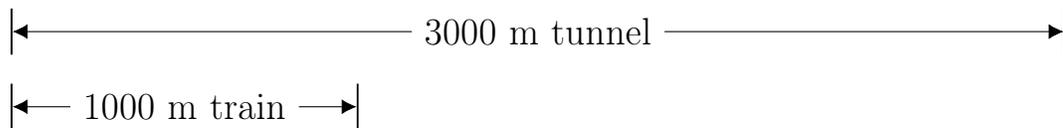
#### Just Passing Through

#### Problem

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#### Solution

A diagram to represent the problem will make it easier to visualize.



At the time the entire train is just inside the tunnel, there is  $3000 - 1000 = 2000$  metres left to travel until the front of the train emerges from the other end. The train has to travel 2000 metres in 30 seconds. We can calculate the speed of the train by dividing the distance travelled by the time required to travel the distance.

The speed of the train is  $2000 \text{ m} \div 30 \text{ seconds} = \frac{200}{3} \text{ m/s}$ .

Now our task is to convert from m/s to km/h. We will do this in two steps: first convert metres to kilometres and then convert seconds to hours.

$$(1) \quad \frac{200 \text{ m}}{3 \text{ s}} = \frac{200 \text{ m}}{3 \text{ s}} \times \frac{1 \text{ km}}{1000 \text{ m}} = \frac{200 \text{ km}}{3000 \text{ s}} = \frac{1 \text{ km}}{15 \text{ s}}$$

$$(2) \quad \frac{1 \text{ km}}{15 \text{ s}} = \frac{1 \text{ km}}{15 \text{ s}} \times \frac{60 \text{ s}}{1 \text{ min}} \times \frac{60 \text{ min}}{1 \text{ h}} = \frac{3600 \text{ km}}{15 \text{ h}} = \frac{240 \text{ km}}{1 \text{ h}}$$

The train is travelling at a speed of 240 km/h.



## Problem of the Week

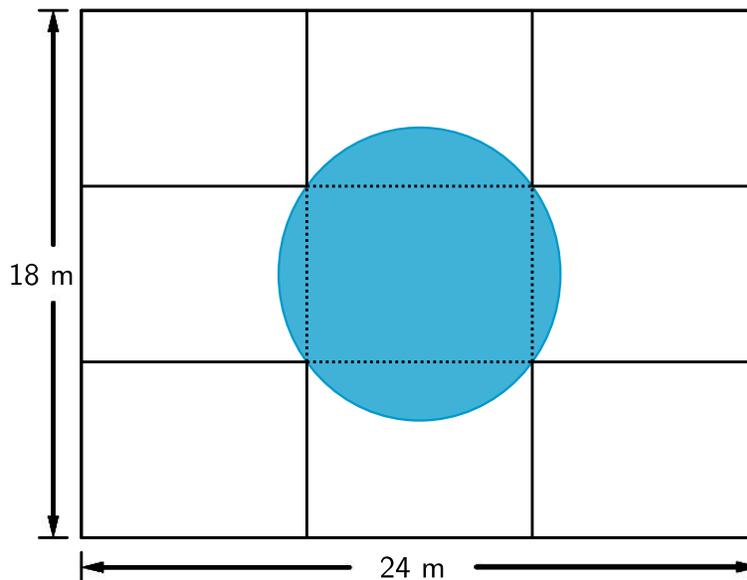
### Problem C

### Making a Splash!

A family has decided that they want to have a pool installed in their backyard. The city they live in has a by-law which states, “No pool may occupy more than 20% of the total area of the backyard in which it is to be installed.”

Their backyard is rectangular and measures 24 m by 18 m. The family creates a design in which they divide their backyard into a 3 by 3 grid of nine identical rectangles. The pool will be circular with the circumference of the pool passing through the four vertices of the middle rectangle. The middle rectangle will be completely covered by the pool. The distance across the pool through its centre will equal the length of the diagonal of the middle rectangle. Their plan is illustrated on the diagram below.

Should their plan be approved by the city?

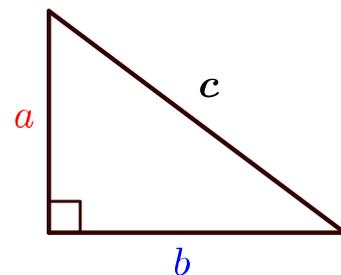


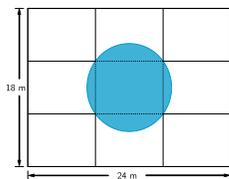
You may find the following useful:

The *Pythagorean Theorem* states, “In a right-angled triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides.”

In the right-angled triangle shown,  $c$  is the hypotenuse and

$$c^2 = a^2 + b^2$$





## Problem of the Week

### Problem C and Solution

#### Making a Splash!

#### Problem

A family has decided that they want to have a pool installed in their backyard. The city they live in has a by-law which states, “No pool may occupy more than 20% of the total area of the backyard in which it is to be installed.”

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#### Solution

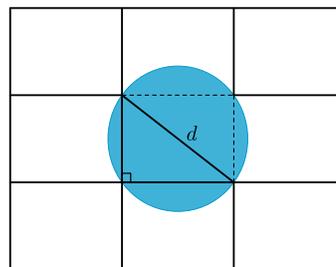
The dimensions of the backyard are 24 m by 18 m, so the area of the entire backyard is  $24 \times 18 = 432 \text{ m}^2$ . To comply with the by-law, the area of the pool cannot be more than 20% or  $\frac{1}{5}$  of the total area of the backyard.

That is, the area cannot exceed  $\frac{1}{5}$  of 432, which is equal to  $\frac{1}{5} \times 432 = 86.4 \text{ m}^2$ .

Since we know the length of the backyard is 24 m and the backyard is 3 rectangles long, the length of one rectangle is  $24 \div 3 = 8 \text{ m}$ . Similarly, since we know the width of the backyard is 18 m and the backyard is 3 rectangles across, the width of one rectangle is  $18 \div 3 = 6 \text{ m}$ .

Two adjacent sides of the middle rectangle and a diagonal form a right-angled triangle. Let  $d$  represent the length of the hypotenuse of the right-angled triangle with sides 8 m and 6 m. We know that  $d$  is also the length of the diameter of the pool. Using the Pythagorean Theorem,

$$\begin{aligned} d^2 &= 8^2 + 6^2 \\ &= 64 + 36 \\ &= 100 \\ d &= \sqrt{100}, \text{ since } d > 0 \\ d &= 10 \text{ m} \end{aligned}$$



Therefore, the diameter of the pool is 10 m, so it follows that the radius of the pool will be  $10 \div 2 = 5 \text{ m}$ .

The area of the pool can be calculated using the formula  $A = \pi r^2$  so  $A = \pi \times (5)^2 = 25\pi \approx 78.5 \text{ m}^2$ . Since  $78.5 < 86.4$ , the area of the pool is less than one-fifth of the total area of the backyard.

Therefore, their plan should be approved.



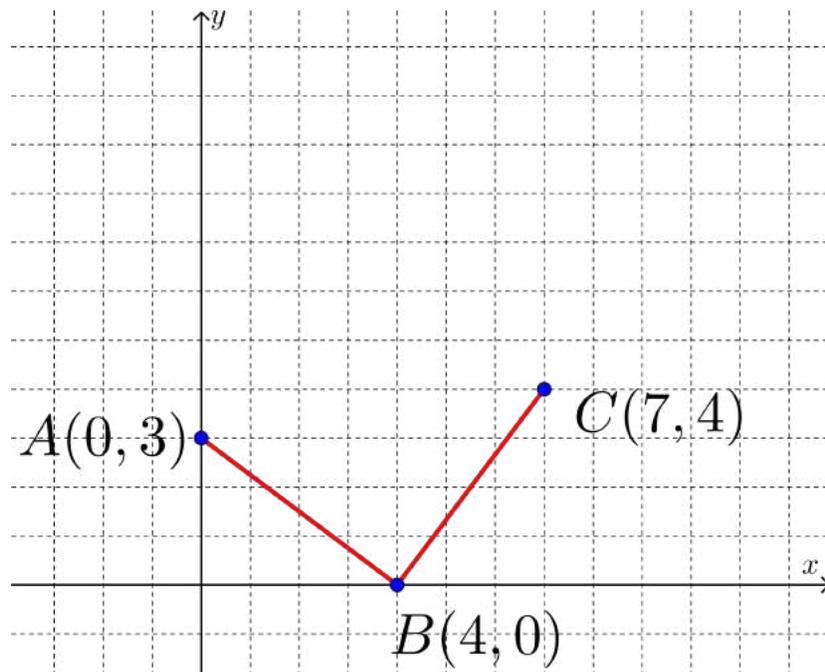
## Problem of the Week

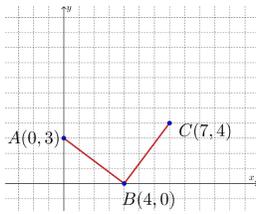
### Problem C

### Penned In

A square enclosure, labelled  $ABCD$ , is sketched out on a piece of graph paper. Three of the vertices of the square  $ABCD$  are located at  $A(0, 3)$ ,  $B(4, 0)$ , and  $C(7, 4)$ .

Determine the area of the square enclosure.





## Problem of the Week

### Problem C and Solution

#### Penned In

#### Problem

A square enclosure, labelled  $ABCD$ , is sketched out on a piece of graph paper. Three of the vertices of the square  $ABCD$  are located at  $A(0, 3)$ ,  $B(4, 0)$ , and  $C(7, 4)$ . Determine the area of the square enclosure.

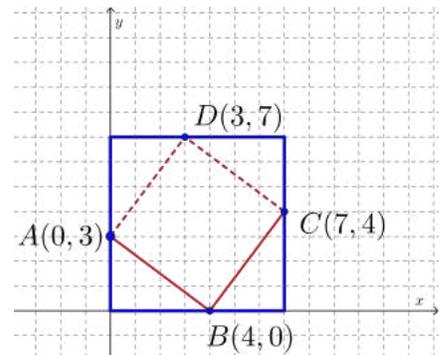
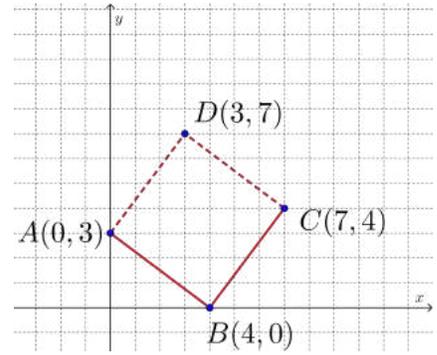
#### Solution

##### Solution 1

In this solution we will determine the area of  $ABCD$  without using the Pythagorean Theorem.

We will first determine the coordinates of the fourth vertex  $D$ . To do so, observe that to get from  $A$  to  $B$ , you would go down 3 units and right 4 units. To get from  $B$  to  $C$ , you move 3 units to the right and then 4 units up. Continuing the pattern, going up 3 units and left 4 units, you get to  $D(3, 7)$ . Continuing, as a check, go left 3 units and down 4 units, and you arrive back at  $A$ . The coordinates of  $D$  are  $(3, 7)$ .

Draw a box with horizontal and vertical sides so that each vertex of the square  $ABCD$  is on one of the sides of the box. This creates a large square with sides of length 7 containing four congruent triangles and square  $ABCD$ . Each of the triangles has a base 4 units long and height 3 units long.



$$\begin{aligned}
 \text{Area } ABCD &= \text{Area of Large Square} - 4 \times \text{Area of One Triangle} \\
 &= \text{Length} \times \text{Width} - 4 \times (\text{Base} \times \text{Height} \div 2) \\
 &= 7 \times 7 - 4 \times (4 \times 3 \div 2) \\
 &= 49 - 4 \times 6 \\
 &= 49 - 24 \\
 &= 25
 \end{aligned}$$

Therefore, the area of the enclosure is 25 units<sup>2</sup>.



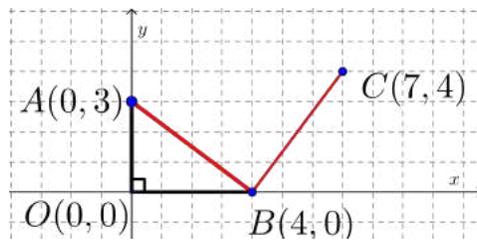
## Solution 2

In this solution we will determine the area of  $ABCD$  using the Pythagorean Theorem.

Since  $ABCD$  is a square, it is only necessary to find the length of one side. We can determine the area by squaring the length of the side.

Label the origin  $O$ .

$OA$ , the distance from the origin to point  $A$  on the  $y$ -axis, is 3 units.  $OB$ , the distance from the origin to point  $B$  on the  $x$ -axis, is 4 units. Since  $A$  lies on the  $y$ -axis and  $B$  lies on the  $x$ -axis,  $OAB$  forms a right-angled triangle.



Using the Pythagorean Theorem in right-angled  $\triangle OAB$ , we can find  $AB^2$  which is  $AB \times AB$ , the area of the square.

$$\begin{aligned} AB^2 &= OA^2 + OB^2 \\ &= 3^2 + 4^2 \\ &= 9 + 16 \\ &= 25 \end{aligned}$$

Therefore, the area of the enclosure is 25 units<sup>2</sup>.

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# Algebra (A)

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## Problem of the Week

### Problem C

### To the Next Level

Maddie Leet is participating in the first round of a math competition. She writes one contest a month in each of the first ten months of the school year. She can earn up to 100 points on each contest. On each of the first five contests she averaged 68 points. On the next three contests she averaged 80 points. In order to advance to the next round of the competition, she must obtain a minimum total of 750 points on the ten contests.

What is the minimum average Maddie requires on the final two contests in order to be able to advance to the next round of the competition?





## Problem of the Week

### Problem C and Solution

### To the Next Level

#### Problem

Maddie Leet is participating in the first round of a math competition. She writes one contest a month in each of the first ten months of the school year. She can earn up to 100 points on each contest. On each of the first five contests she averaged 68 points. On the next three contests she averaged 80 points. In order to advance to the next round of the competition, she must obtain a minimum total of 750 points on the ten contests. What is the minimum average Maddie requires on the final two contests in order to be able to advance to the next round of the competition?

#### Solution

To determine an average of some numbers, we add the numbers together and divide the sum by the number of numbers.

$$\text{Average} = \frac{\text{Sum of Numbers}}{\text{Number of Numbers}}$$

Therefore, to determine the sum of the numbers we multiply their average by the number of numbers.

$$\text{Sum of Numbers} = \text{Average} \times \text{Number of Numbers}$$

$$\text{Total Score for First 5 Contests} = 68 \times 5 = 340$$

$$\text{Total Score for Next 3 Contests} = 80 \times 3 = 240$$

$$\text{Total Score to Move On} = 750$$

$$\text{Total Score Needed for Last Two Contests} = 750 - 340 - 240 = 170$$

$$\text{Average Score for Final 2 Contests} = 170 \div 2 = 85$$

Maddie needs to average 85 points on her last two contests to have a total score of exactly 750 points. Therefore, the minimum average Maddie needs on the final two contests in order to advance to the next round of the math competition is 85 points. If she averages anything less than 85 points she will not move on. If she averages more than 85 points she will obtain a total score over 750 points and will move on to the next round.



## Problem of the Week

### Problem C

#### This Product is a Mystery

The number  $A8$  is a two-digit number with tens digit  $A$  and units (ones) digit 8. Similarly,  $3B$  is a two-digit number with tens digit 3 and units digit  $B$ . When  $A8$  is multiplied by  $3B$ , the result is the four-digit number  $C730$ . That is,

$$\begin{array}{r} A8 \\ \times 3B \\ \hline C730 \end{array}$$



If  $A$ ,  $B$ , and  $C$  are each different digits from 0 to 9, determine the values of  $A$ ,  $B$ , and  $C$ .



## Problem of the Week

### Problem C and Solution

### This Product is a Mystery

#### Problem

The number  $A8$  is a two-digit number with tens digit  $A$  and units (ones) digit 8. Similarly,  $3B$  is a two-digit number with tens digit 3 and units digit  $B$ .

When  $A8$  is multiplied by  $3B$ , the result is the four-digit number  $C730$ . If  $A$ ,  $B$ , and  $C$  are each different digits from 0 to 9, determine the values of  $A$ ,  $B$ , and  $C$ .

#### Solution

In a multiplication question there are three parts: the *multiplier*, *multiplicand* and *product*. In our problem,  $A8$  is the multiplier,  $3B$  is the multiplicand, and  $C730$  is the product.

The units digit of the product  $C730$  is 0. The units digit of a product is equal to the units digit of the result obtained by multiplying the units digits of the multiplier and multiplicand.

So  $8 \times B$  must equal a number with units digit 0. The only choices for  $B$  are 0 and 5, since no other single digit multiplied by 8 produces a number ending in zero.

However, if  $B = 0$ , the units digit of the product is 0 and the remaining three digits of the product,  $C73$ , are produced by multiplying  $3 \times A8$ . But  $3 \times A8$  produces a number ending in 4, not 3 as required. Therefore  $B \neq 0$  and  $B$  must equal 5. So the multiplicand is 35.

Since  $A8$  is a two-digit number, the largest possible value for  $A$  is 9. Since  $98 \times 35 = 3430$ , the largest possible value of  $C$  is 3. Also, the product  $C730$  is a four-digit number so  $C \neq 0$ .

Therefore, the only possible values for  $C$  are 1, 2, and 3. We will examine each possibility for  $C$ .

If  $C = 1$ , then  $C730$  becomes 1730. We want  $A8 \times 35 = 1730$ . Alternatively, we want  $1730 \div 35 = A8$ . But  $1730 \div 35 \doteq 49.4$ , which is not a whole number. Therefore,  $C \neq 1$ .

If  $C = 2$ , then  $C730$  becomes 2730. We want  $A8 \times 35 = 2730$ . Alternatively, we want  $2730 \div 35 = A8$ . Since  $2730 \div 35 = 78$ , which is a whole number. Therefore,  $C = 2$  produces a valid value for  $A$ , namely  $A = 7$ .

If  $C = 3$ , then  $C730$  becomes 3730. We want  $A8 \times 35 = 3730$ . Alternatively, we want  $3730 \div 35 = A8$ . But  $3730 \div 35 \doteq 106.6$ , which is not a whole number. Therefore,  $C \neq 3$ .

We have examined every valid possibility for  $C$  and found only one solution. Therefore,  $A = 7$ ,  $B = 5$  and  $C = 2$  is the only valid solution. We can easily verify that  $78 \times 35 = 2730$ .

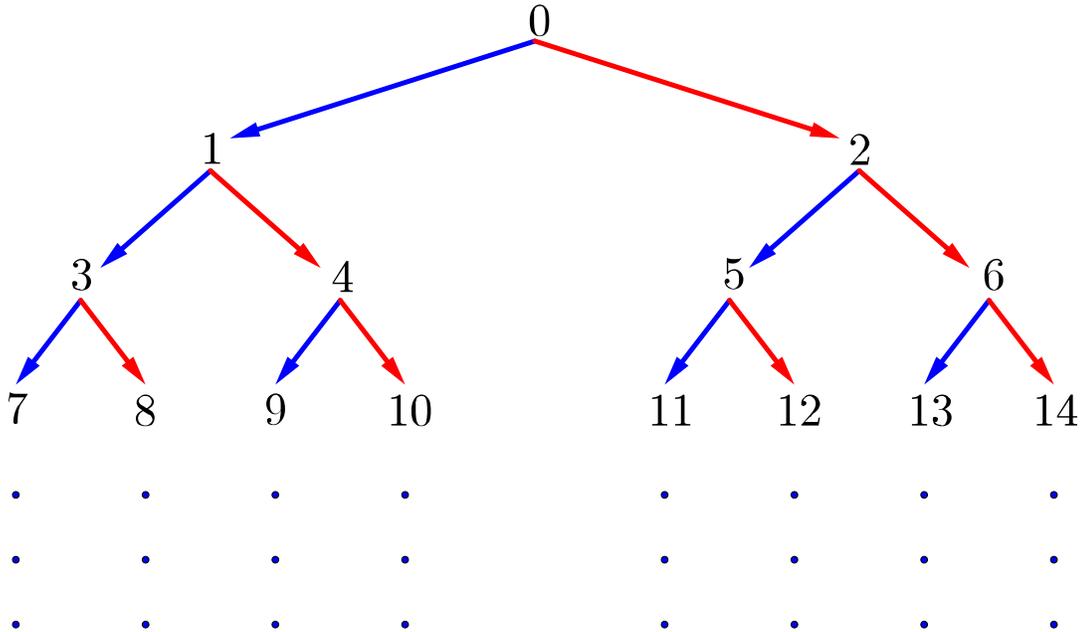


# Problem of the Week

## Problem C

### Branching Out

Consider the following number tree.



In this number tree, the integers greater than or equal to 0 are written out in increasing order, with the top row containing one integer and every row after containing twice as many integers as the row above it.

If row 1 contains the integer 0, what is the fifth number in row 10?



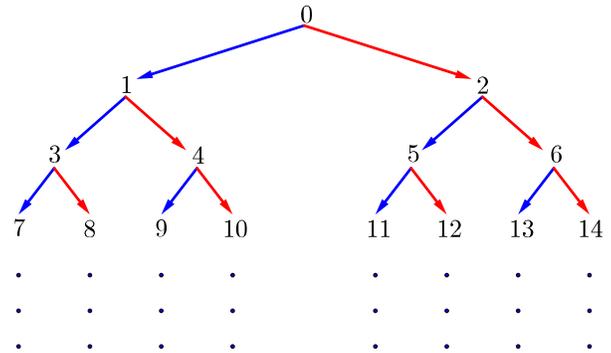
# Problem of the Week

## Problem C and Solution

### Branching Out

#### Problem

Consider the number tree shown to the right. In this number tree, the integers greater than or equal to 0 are written out in increasing order, with the top row containing one integer and every row after containing twice as many integers as the row above it. If row 1 contains the integer 0, what is the fifth number in row 10?



#### Solution

##### Solution 1

One approach to solving the problem would be to write out the first 10 rows of the chart and read off the fifth number in row 10. You would discover that the fifth integer in row 10 is 515. This solution may “work” in this example but it is certainly not ideal. It would not be practical if you were asked for the seventh integer in row 50.

##### Observations

There are many patterns in the tree. The solutions provided below will look at some of the different patterns which can be used to solve the problem.

##### Solution 2

Row 1 contains 1 integer, row 2 contains 2 integers, row 3 contains 4 integers, and row 4 contains 8 integers. Each new row in the tree has twice as many integers as the previous row.

Using this, we could find the number of integers in the first 9 rows of the chart. There are

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 = 511$$

integers in the first 9 rows of the chart. There are 511 integers in the 9 rows, but the first integer in the chart is 0. Therefore, it follows that the last integer in the 9th row is 1 less than the number of integers in 9 rows. That is, the last integer in row 9 is 510. Therefore, the first integer in row 10 is 511. We can easily count to the fifth spot to obtain 515, as above.

**Solution 3**

This solution is similar to Solution 2, but only looks at the rightmost number in each row.

To get from the top number to the rightmost number in row 2, add 2. To get from the rightmost number in row 2 to the rightmost number in row 3, add 4. To get from the rightmost number in row 3 to the rightmost number in row 4, add 8. The numbers that are added correspond to the number of integers in the next row. We will find the rightmost number in row 9 and then add 1 to get the leftmost number in row 10. The rightmost number in row 9 is:

$$0 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 = 510$$

This tells us that the leftmost number in row 10 is 511. Therefore, the fifth number in row 10 is 515.

**Solution 4**

Since each row after the first has twice as many integers as the row above, there is some connection to powers of 2 in the problem. The following table shows the row number, the rightmost number in that row, the power of 2 with the row number as the exponent, and the connection between this power and the last number in the row.

Row Number	Rightmost Number in Row	Power of 2	Connection
1	0	$2^1 = 2$	$2^1 - 2 = 2 - 2 = 0$
2	2	$2^2 = 4$	$2^2 - 2 = 4 - 2 = 2$
3	6	$2^3 = 8$	$2^3 - 2 = 8 - 2 = 6$
4	14	$2^4 = 16$	$2^4 - 2 = 16 - 2 = 14$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n$	??	$2^n$	$2^n - 2 = ??$

It would appear that the last number in row 5 should be  $2^5 - 2 = 32 - 2 = 30$ . We could write out the fifth row to confirm that this is correct.

It would also appear that the last number in row  $n$  should be  $2^n - 2$ . By recognizing the pattern, we predict that the last number in row 9 should be  $2^9 - 2 = 512 - 2 = 510$ . We know from earlier solutions that this is correct.

Therefore, the first number in row 10 is 511 and the fifth number in row 10 is 515.

It should be noted that this relationship works for all of the rows we have sampled, but we have not proven that it is true in general. You will have to wait for some higher mathematics to be able to prove that this is true in general.

The pattern used in this solution is not an obvious one, but by discovering it the solution became fairly straightforward. In fact, if we accept the result as true, then we can quickly determine the value of the rightmost integer in any row with a simple calculation.



## Problem of the Week

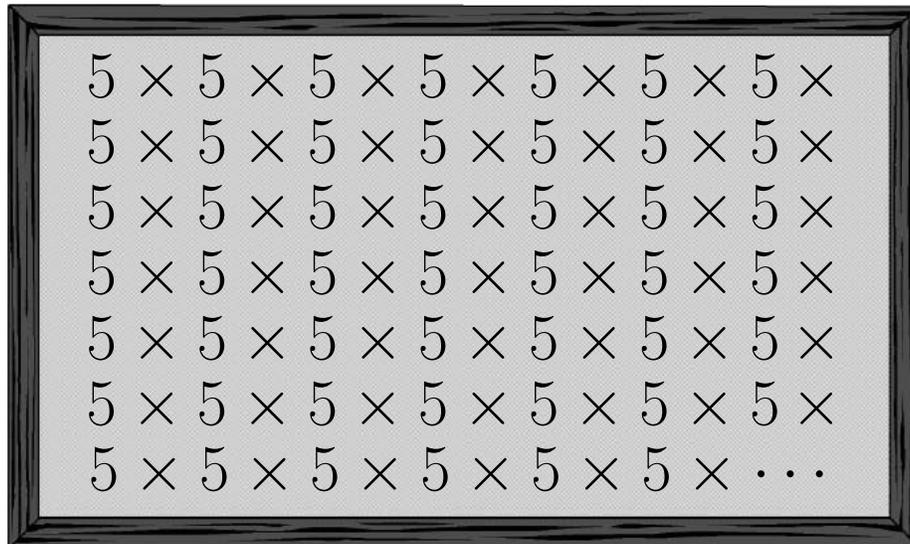
### Problem C

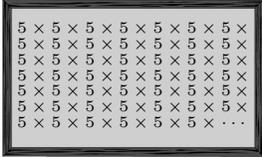
#### More Power to You

In mathematics we like to write expressions concisely. For example, we will often write the expression  $5 \times 5 \times 5 \times 5$  as  $5^4$ . The lower number 5 is called the base, the raised 4 is called the exponent, and the whole expression  $5^4$  is called a power.

So  $5^3$  means  $5 \times 5 \times 5$  and is equal to 125.

What are the last three digits in the integer equal to  $5^{2020}$ ?





## Problem of the Week

### Problem C and Solution

#### More Power to You

#### Problem

In mathematics we like to write expressions concisely. For example, we will often write the expression  $5 \times 5 \times 5 \times 5$  as  $5^4$ . The lower number 5 is called the base, the raised 4 is called the exponent, and the whole expression  $5^4$  is called a power. So  $5^3$  means  $5 \times 5 \times 5$  and is equal to 125. What are the last three digits in the integer equal to  $5^{2020}$ ?

#### Solution

Let's start by examining the last three digits of various powers of 5.

$$5^1 = \quad \mathbf{005}$$

$$5^2 = \quad \mathbf{025}$$

$$5^3 = \quad \mathbf{125}$$

$$5^4 = \quad \mathbf{625}$$

$$5^5 = \quad \mathbf{3125}$$

$$5^6 = \quad \mathbf{15\,625}$$

$$5^7 = \quad \mathbf{78\,125}$$

$$5^8 = \quad \mathbf{390\,625}$$

Notice that there is a pattern for the last three digits after the first two powers of 5. For every odd integer exponent greater than 2, the last three digits are "125". For every even integer exponent greater than 2, the last three digits are "625". If the pattern continues, then  $5^9$  will end "125" since the exponent 9 is odd and  $5^{10}$  will end "625" since the exponent 10 is even. This is easily verified since  $5^9 = 1\,953\,125$  and  $5^{10} = 9\,765\,625$ .

We can easily justify why this pattern continues. If a power ends in "125", then the last 3 digits of the next power are the same as the last three digits of the product  $125 \times 5 = 625$ . That is, the last three digits of the next power are "625". If a power ends in "625", then the last 3 digits of the next power are the same as the last three digits of the product  $625 \times 5 = 3125$ . That is, the last three digits of the next power are "125".

For  $5^{2020}$ , the exponent 2020 is greater than 2 and is an even number.

Therefore, the last three digits of  $5^{2020}$  are 625.

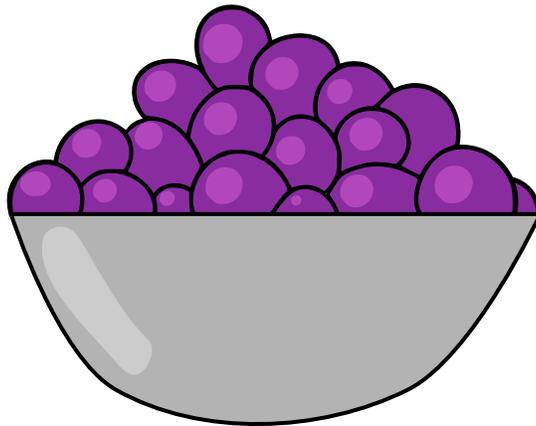


# Problem of the Week

## Problem C

### A Grape Problem

There are several bowls containing various amounts of grapes on a table. When 12 of the bowls each had 8 more grapes added to them, the mean (average) number of grapes per bowl increased by 6. How many bowls of grapes are on the table?





## Problem of the Week

### Problem C and Solution

#### A Grape Problem

#### Problem

There are several bowls containing various amounts of grapes on a table. When 12 of the bowls each had 8 more grapes added to them, the mean (average) number of grapes per bowl increased by 6. How many bowls of grapes are on the table?

#### Solution

##### Solution 1:

The mean number of grapes per bowl is equal to the total number of grapes divided by the number of bowls. So the mean number of grapes per bowl increases by 6 if the total number of grapes added divided by the number of bowls is equal to 6.

If 12 of the bowls each had 8 more grapes added to them, then the total number of grapes added is  $12 \times 8 = 96$ .

Thus, 96 divided by the number of bowls equals 6. Since  $96 \div 6 = 16$ , there are 16 bowls.

##### Solution 2:

This solution uses variables to represent the unknown values.

We are told that the mean number of grapes per bowl increased by 6. We can write this as follows.

$$\text{old mean} + 6 = \text{new mean}$$

The mean is equal to the total number of grapes divided by the number of bowls. Let  $b$  be the number of bowls. Let  $g$  be the total number of grapes that were *originally* in the bowls. If 12 of the bowls each had 8 more grapes added to them, then the total number of grapes added is  $12 \times 8 = 96$ . Now we can rewrite our equation above using variables.

$$\frac{g}{b} + 6 = \frac{g + 96}{b}$$

We can separate the fraction on the right side.

$$\frac{g}{b} + 6 = \frac{g}{b} + \frac{96}{b}$$

Now if we subtract  $\frac{g}{b}$  from both sides of the equation, we are left with an equation that has only one variable. We can then solve this equation.

$$6 = \frac{96}{b}$$

$$\text{Multiply both sides by } b: \quad 6b = 96$$

$$\text{Divide both sides by } 6: \quad b = \frac{96}{6}$$

$$b = 16$$

Therefore, there are 16 bowls.



## Problem of the Week

### Problem C

#### How It Ends

The product of the positive integers 1 to 4 is

$$4 \times 3 \times 2 \times 1 = 24$$

and can be written in an abbreviated form as  $4!$ . We say “4 *factorial*”. So  $4! = 24$ .

The product of the positive integers 1 to 16 is

$$16 \times 15 \times 14 \times \cdots \times 3 \times 2 \times 1$$

and can be written in an abbreviated form as  $16!$ . We say “16 *factorial*”.

The  $\cdots$  represents the product of all the missing integers between 14 and 3.

In general, the product of the positive integers 1 to  $n$  is  $n!$ . Note that  $1! = 1$ .

Determine the tens digit and units (ones) digit of the sum

$$1! + 2! + 3! + \cdots + 2019! + 2020! + 2021!$$





## Problem of the Week

### Problem C and Solution

#### How It Ends

#### Problem

The product of the positive integers 1 to 4 is  $4 \times 3 \times 2 \times 1 = 24$  and can be written in an abbreviated form as  $4!$ . We say “4 *factorial*”. So  $4! = 24$ .

The product of the positive integers 1 to 16 is  $16 \times 15 \times 14 \times \cdots \times 3 \times 2 \times 1$  and can be written in an abbreviated form as  $16!$ . We say “16 *factorial*”.

The  $\cdots$  represents the product of all the missing integers between 14 and 3.

In general, the product of the positive integers 1 to  $n$  is  $n!$ . Note that  $1! = 1$ .

Determine the tens digit and units (ones) digit of the sum

$$1! + 2! + 3! + \cdots + 2019! + 2020! + 2021!$$

#### Solution

At first glance seems like there is a great deal of work to do. However, by examining several factorials, we will discover otherwise.

$$1! = 1$$

$$2! = 2 \times 1 = 2$$

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Now  $6! = 6 \times (5 \times 4 \times 3 \times 2 \times 1) = 6 \times 5! = 6(120) = 720$ ,

$$7! = 7 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 7 \times 6! = 7(720) = 5040,$$

$$8! = 8 \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 8 \times 7! = 8(5040) = 40\,320,$$

$$9! = 9 \times (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 9 \times 8! = 9(40\,320) = 362\,880, \text{ and}$$

$$10! = 10 \times (9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 10 \times 9! = 10(362\,880) = 3\,628\,800.$$

An interesting observation surfaces,  $9! = 9 \times 8!$ ,  $10! = 10 \times 9!$ ,  $11! = 11 \times 10!$ , and so on.

Furthermore, the last two digits of  $10!$  are 00. Every factorial above  $10!$  will also end with 00 since multiplying an integer that ends with 00 by another integer produces an integer product that ends in 00. So all factorials above  $10!$  will end with 00 and will not change the tens digit or the units digit in the required sum. We can determine the last two digits of the required sum by adding the last two digits of each of the factorials from  $1!$  to  $9!$ .

The sum of the last two digits of  $1!$  to  $9!$  will equal

$$1 + 2 + 6 + 24 + 20 + 20 + 40 + 20 + 80 = 213.$$

Therefore, for the sum  $1! + 2! + 3! + \cdots + 2019! + 2020! + 2021!$ , the tens digit will be 1 and the units (ones) digit will be 3. From what we have done, we do not know the hundreds digit.

**For Further Thought:** If we had only been interested in the units digit in the required sum, how many factorials would we need to calculate? If we wanted to know the last three digits, how many more factorials would be required?



## Problem of the Week

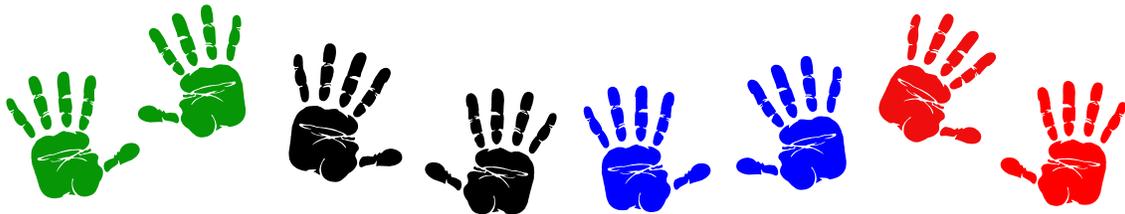
### Problem C

#### Painting the Way

Alexia, Benito, and Carmen won a team art competition at their local park. As part of their prize, they can paint the 5 km (5000 m) paved path through the park any way they like.

They decided Alexia will paint the first 70 m, then Benito will paint the next 15 m, then Carmen will paint the next 35 m. They will keep repeating this pattern until they reach the end of the 5 km path.

What percentage of the path will each person paint?





## Problem of the Week

### Problem C and Solution

#### Painting the Way

**Problem**

Alexia, Benito, and Carmen won a team art competition at their local park. As part of their prize, they can paint the 5 km (5000 m) paved path through the park any way they like. They decided Alexia will paint the first 70 m, then Benito will paint the next 15 m, then Carmen will paint the next 35 m. They will keep repeating this pattern until they reach the end of the 5 km path. What percentage of the path will each person paint?

**Solution**

After each person paints their first section, they will have covered  $70 + 15 + 35 = 120$  m in total. We will call this one cycle. The number of cycles in 5000 m is

$$\frac{5000}{120} = \frac{125}{3} = 41\frac{2}{3}.$$

So, there are 41 complete cycles and  $\frac{2}{3}$  of another cycle. The total distance covered in 41 cycles is  $41 \times 120 = 4920$  m. That means there are  $5000 - 4920 = 80$  m remaining. Alexia would paint the first 70 m, leaving the last 10 m for Benito to paint.

We can organize this information in a table to calculate the total distance each person will paint.

	Distance Painted by Each Person (m)		
	Alexia	Benito	Carmen
<b>First 41 cycles</b>	$41 \times 70 = 2870$	$41 \times 15 = 615$	$41 \times 35 = 1435$
<b>Last partial cycle</b>	70	10	0
<b>Total</b>	$2870 + 70 = 2940$	$615 + 10 = 625$	1435

We can now calculate the percentage of the path each person will paint.

$$\text{Alexia: } \frac{2940}{5000} \times 100\% = 58.8\%$$

$$\text{Benito: } \frac{625}{5000} \times 100\% = 12.5\%$$

$$\text{Carmen: } \frac{1435}{5000} \times 100\% = 28.7\%$$

Therefore, Alexia will paint 58.8% of the path, Benito will paint 12.5% of the path, and Carmen will paint 28.7% of the path.



## Problem of the Week

### Problem C

#### Greta's New Gig

Greta currently works 45 hours per week and earns a weekly salary of \$729. She will soon be starting a new job where her salary will be increased by 10% and her hours reduced by 10%.

How much more will she be earning per hour at her new job?





## Problem of the Week

### Problem C and Solution

#### Greta's New Gig

#### Problem

Greta currently works 45 hours per week and earns a weekly salary of \$729. She will soon be starting a new job where her salary will be increased by 10% and her hours reduced by 10%. How much more will she be earning per hour at her new job?

#### Solution

##### Solution 1

To calculate how much Greta earns per hour (i.e. her hourly rate of pay), divide her weekly salary by the number of hours worked.

Greta's old hourly rate of pay is  $\$729 \div 45 \text{ h} = \$16.20/\text{h}$ .

$$\begin{aligned}\text{New Weekly Salary} &= \text{Old Weekly Salary} + 10\% \text{ of Old Weekly Salary} \\ &= \$729 + 0.1 \times \$729 \\ &= \$729 + \$72.90 \\ &= \$801.90\end{aligned}$$

$$\begin{aligned}\text{New Number of Hours Worked} &= \text{Old Hours Worked} - 10\% \text{ of Old Hours Worked} \\ &= 45 \text{ h} - 0.1 \times 45 \text{ h} \\ &= 45 \text{ h} - 4.5 \text{ h} \\ &= 40.5 \text{ h}\end{aligned}$$

Greta's new hourly rate of pay is  $\$801.90 \div 40.5 \text{ h} = \$19.80/\text{h}$ .

The change in her hourly rate of pay is  $\$19.80/\text{h} - \$16.20/\text{h} = \$3.60/\text{h}$ .

Therefore, Greta will be earning \$3.60/h more at her new job.

##### Solution 2

In the second solution we will use a more concise calculation. Greta's new weekly salary is 10% more than her old weekly salary. So Greta will earn 110% of her old weekly salary. Greta's hours will be reduced by 10%, so her new hours will be 90% of her old hours. To calculate her change in hourly rate we can take her new hourly rate and subtract her old hourly rate.

$$\begin{aligned}\text{Change in Hourly Rate} &= \text{New Hourly Rate} - \text{Old Hourly Rate} \\ &= \text{New Salary} \div \text{New Hours Worked} - \text{Old Salary} \div \text{Old Hours Worked} \\ &= (\$729 \times 1.10) \div (45 \times 0.9) - \$729 \div 45 \\ &= \$801.90 \div 40.5 - \$729 \div 45 \\ &= \$19.80/\text{h} - \$16.20/\text{h} \\ &= \$3.60/\text{h}\end{aligned}$$

Therefore, Greta will be earning \$3.60/h more at her new job.

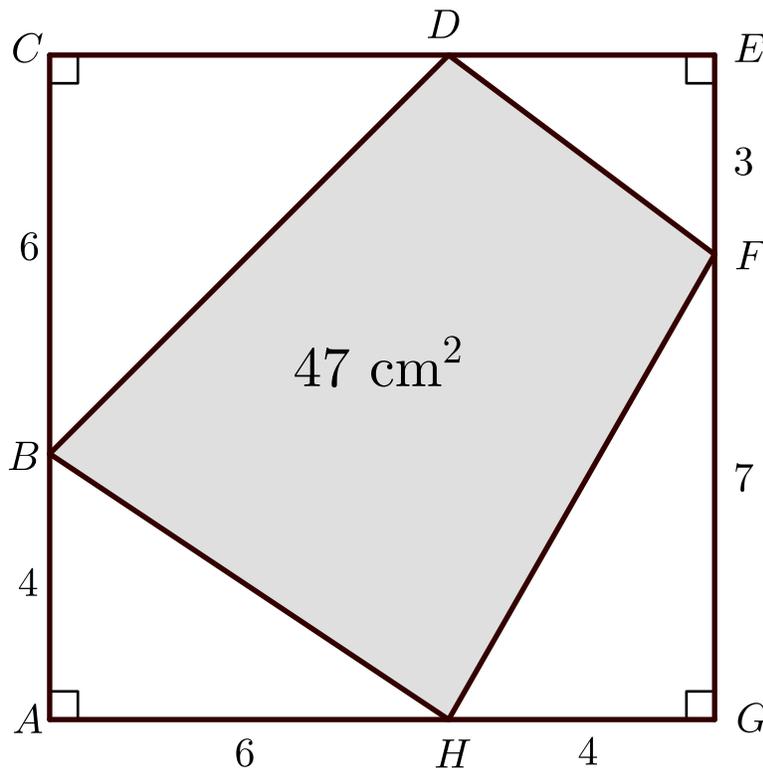


**Problem of the Week**  
**Problem C**  
**Locate the Fourth Vertex**

Quadrilateral  $BDFH$  is constructed so that each vertex is on a different side of square  $ACEG$ . Vertex  $B$  is on side  $AC$  so that  $AB = 4$  cm and  $BC = 6$  cm. Vertex  $F$  is on  $EG$  so that  $EF = 3$  cm and  $FG = 7$  cm. Vertex  $H$  is on  $GA$  so that  $GH = 4$  cm and  $HA = 6$  cm. The area of quadrilateral  $BDFH$  is  $47$  cm<sup>2</sup>.

The fourth vertex of quadrilateral  $BDFH$ , labelled  $D$ , is located on side  $CE$  so that the lengths of  $CD$  and  $DE$  are both positive integers.

Determine the lengths of  $CD$  and  $DE$ .





## Problem of the Week Problem C and Solution Locate the Fourth Vertex

### Problem

Quadrilateral  $BDFH$  is constructed so that each vertex is on a different side of square  $ACEG$ . Vertex  $B$  is on side  $AC$  so that  $AB = 4$  cm and  $BC = 6$  cm. Vertex  $F$  is on  $EG$  so that  $EF = 3$  cm and  $FG = 7$  cm. Vertex  $H$  is on  $GA$  so that  $GH = 4$  cm and  $HA = 6$  cm. The area of quadrilateral  $BDFH$  is  $47$  cm<sup>2</sup>.

The fourth vertex of quadrilateral  $BDFH$ , labelled  $D$ , is located on side  $CE$  so that the lengths of  $CD$  and  $DE$  are both positive integers.

Determine the lengths of  $CD$  and  $DE$ .

### Solution

#### Solution 1

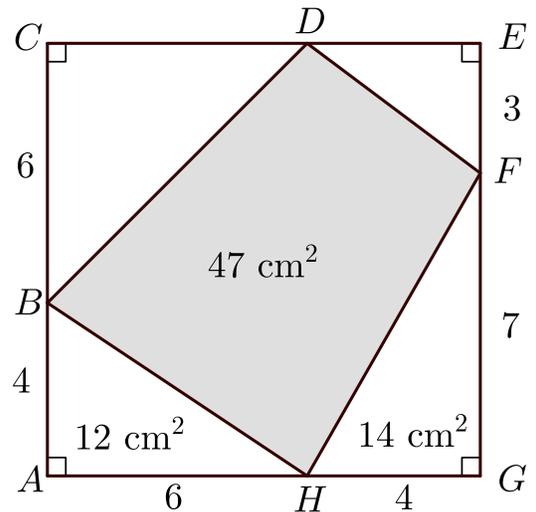
Since  $ACEG$  is a square and the length of  $AG = AH + HG = 6 + 4 = 10$  cm, then the length of each side of the square is 10 cm and the area is  $10 \times 10 = 100$  cm<sup>2</sup>.

We can determine the area of triangles  $BAH$  and  $FGH$  using the formula  $\text{area} = \frac{\text{base} \times \text{height}}{2}$ .

In  $\triangle BAH$ , since  $BA$  is perpendicular to  $AH$ , we can use  $BA$  as the height and  $AH$  as the base. The area of  $\triangle BAH$  is  $\frac{6 \times 4}{2} = 12$  cm<sup>2</sup>.

In  $\triangle FGH$ , since  $FG$  is perpendicular to  $GH$ , we can use  $FG$  as the height and  $GH$  as the base. The area of  $\triangle FGH$  is  $\frac{4 \times 7}{2} = 14$  cm<sup>2</sup>.

The area of  $\triangle BCD$  plus the area of  $\triangle FED$  must be the total area minus the three known areas. That is,  
 $\text{Area } \triangle BCD + \text{Area } \triangle FED = 100 - 12 - 47 - 14 = 27$  cm<sup>2</sup>.



$CD$  and  $DE$  are both positive integers and  $CD + DE = 10$ . We will systematically check all possible values for  $CD$  and  $DE$  to determine the values which produce the correct area.

$CD$	$DE$	Area $\triangle BCD$	Area $\triangle FED$	Area $\triangle BCD + \text{Area } \triangle FED$
1	9	$1 \times 6 \div 2 = 3$	$9 \times 3 \div 2 = 13.5$	$3 + 13.5 = 16.5 \neq 27$
2	8	$2 \times 6 \div 2 = 6$	$8 \times 3 \div 2 = 12$	$6 + 12 = 18 \neq 27$
3	7	$3 \times 6 \div 2 = 9$	$7 \times 3 \div 2 = 10.5$	$9 + 10.5 = 19.5 \neq 27$
4	6	$4 \times 6 \div 2 = 12$	$6 \times 3 \div 2 = 9$	$12 + 9 = 21 \neq 27$
5	5	$5 \times 6 \div 2 = 15$	$5 \times 3 \div 2 = 7.5$	$15 + 7.5 = 22.5 \neq 27$
6	4	$6 \times 6 \div 2 = 18$	$4 \times 3 \div 2 = 6$	$18 + 6 = 24 \neq 27$
7	3	$7 \times 6 \div 2 = 21$	$3 \times 3 \div 2 = 4.5$	$21 + 4.5 = 25.5 \neq 27$
8	2	$8 \times 6 \div 2 = 24$	$2 \times 3 \div 2 = 3$	$24 + 3 = 27$
9	1	$9 \times 6 \div 2 = 27$	$1 \times 3 \div 2 = 1.5$	$27 + 1.5 = 28.5 \neq 27$

Therefore, when  $CD = 8$  cm and  $DE = 2$  cm, the area of quadrilateral  $BDFH$  is  $47$  cm<sup>2</sup>.

The second solution is more algebraic and will produce a solution for any lengths of  $CD$  and  $DE$  between 0 and 10 cm.



**Solution 2**

This solution begins the same as Solution 1. Algebra is introduced to complete the solution.

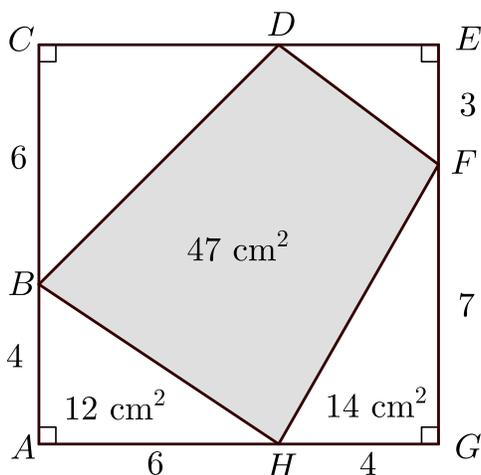
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Let the length of  $CD$  be  $n$  cm. Then the length of  $DE$  is  $(10 - n)$  cm.

The area of  $\triangle BCD$  is  $\frac{BC \times CD}{2} = \frac{6 \times n}{2} = 3n$ .

The area of  $\triangle FED$  is  $\frac{FE \times DE}{2} = \frac{3 \times (10 - n)}{2} = \frac{10 - n + 10 - n + 10 - n}{2} = \frac{30 - 3n}{2}$ .

Therefore,

$$\text{Area } \triangle BCD + \text{Area } \triangle FED = 27$$

$$3n + \frac{30 - 3n}{2} = 27$$

Multiplying both sides by 2:  $6n + 30 - 3n = 54$

$$3n + 30 = 54$$

$$3n = 24$$

$$n = 8$$

Therefore, the length of  $CD$  is 8 cm and the length of  $DE$  is 2 cm.

The algebra presented in Solution 2 may not be familiar to all students at this level.



## Problem of the Week

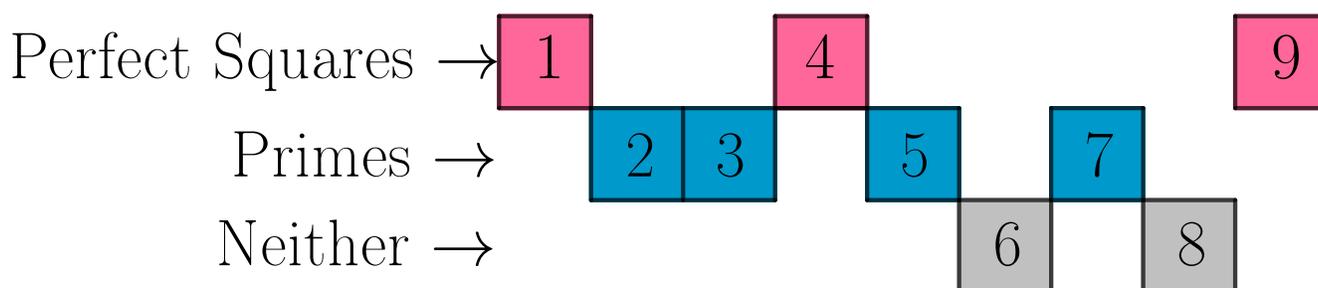
### Problem C

#### These Primes are Squares!

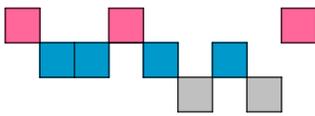
The number 7 has only two positive factors, 1 and itself. A positive integer greater than 1 whose only positive factors are 1 and itself is said to be *prime*.

A *perfect square* is an integer created by multiplying an integer by itself. The number 25 is a perfect square since it is  $5 \times 5$  or  $5^2$ .

Determine the smallest perfect square that has three different prime numbers as factors.



EXTENSION: Determine all perfect squares less than 10 000 that have three different prime numbers as factors.



## Problem of the Week

### Problem C and Solution

### These Primes are Squares!

#### Problem

The number 7 has only two positive factors, 1 and itself. A positive integer greater than 1 whose only positive factors are 1 and itself is said to be *prime*.

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Determine the smallest perfect square that has three different prime numbers as factors.

EXTENSION: Determine all perfect squares less than 10 000 that have three different prime numbers as factors.

#### Solution

The problem itself is not very difficult once you determine what it is asking. Let's begin by examining some perfect squares.

The numbers 4 and 9 are both perfect squares that have only one prime number as a factor,  $4 = 2^2$  and  $9 = 3^2$ . The number 36 is a perfect square since  $36 = 6^2$ . However, the number  $6 = 2 \times 3$  so  $36 = (2 \times 3)^2 = 2 \times 3 \times 2 \times 3 = 2^2 \times 3^2$ . So 36 is the product of the squares of two different prime numbers.

Notice that when we write a perfect square as a product of prime factors, each prime factor appears an even number of times in the product. This is because the perfect square is created by multiplying an integer by itself, so all of the prime factors of the integer appear twice.

To create the smallest perfect square with three different prime factors, we should choose the three smallest prime numbers, namely 2, 3, and 5. If we include any primes larger than these, then their product will be larger and so the perfect square will be larger. To make our number a perfect square, we should have each prime factor appear twice. This gives us  $2^2 \times 3^2 \times 5^2 = 4 \times 9 \times 25 = 900$ . It should be noted that  $900 = 30^2 = (2 \times 3 \times 5)^2$ .

Therefore, the smallest perfect square with three different prime numbers as factors is 900.

#### EXTENSION ANSWER:

Since perfect squares have an even number of each prime factor in their factorization, we can find all the perfect squares less than 10 000 with three different prime factors as numbers by systematically trying different combinations of prime numbers. The table below lists all the perfect squares less than 10 000 with three different prime factors.

$2^2 \times 3^2 \times 5^2 = 900$	$2^2 \times 2^2 \times 3^2 \times 5^2 = 3600$	$2^2 \times 3^2 \times 3^2 \times 5^2 = 8100$
$2^2 \times 3^2 \times 7^2 = 1764$	$2^2 \times 2^2 \times 3^2 \times 7^2 = 7056$	
$2^2 \times 3^2 \times 11^2 = 4356$		
$2^2 \times 3^2 \times 13^2 = 6084$		
$2^2 \times 5^2 \times 7^2 = 4900$		



## Problem of the Week

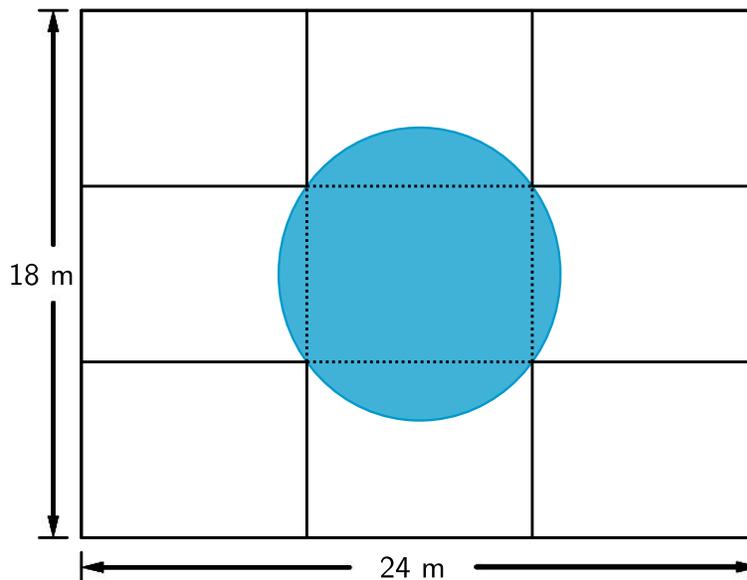
### Problem C

### Making a Splash!

A family has decided that they want to have a pool installed in their backyard. The city they live in has a by-law which states, “No pool may occupy more than 20% of the total area of the backyard in which it is to be installed.”

Their backyard is rectangular and measures 24 m by 18 m. The family creates a design in which they divide their backyard into a 3 by 3 grid of nine identical rectangles. The pool will be circular with the circumference of the pool passing through the four vertices of the middle rectangle. The middle rectangle will be completely covered by the pool. The distance across the pool through its centre will equal the length of the diagonal of the middle rectangle. Their plan is illustrated on the diagram below.

Should their plan be approved by the city?

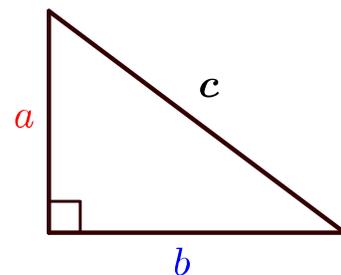


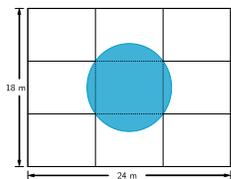
You may find the following useful:

The *Pythagorean Theorem* states, “In a right-angled triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides.”

In the right-angled triangle shown,  $c$  is the hypotenuse and

$$c^2 = a^2 + b^2$$





## Problem of the Week

### Problem C and Solution

#### Making a Splash!

#### Problem

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#### Solution

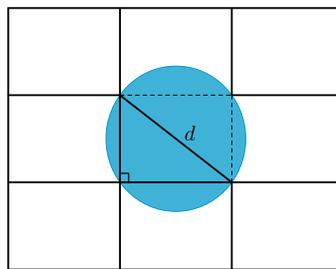
The dimensions of the backyard are 24 m by 18 m, so the area of the entire backyard is  $24 \times 18 = 432 \text{ m}^2$ . To comply with the by-law, the area of the pool cannot be more than 20% or  $\frac{1}{5}$  of the total area of the backyard.

That is, the area cannot exceed  $\frac{1}{5}$  of 432, which is equal to  $\frac{1}{5} \times 432 = 86.4 \text{ m}^2$ .

Since we know the length of the backyard is 24 m and the backyard is 3 rectangles long, the length of one rectangle is  $24 \div 3 = 8 \text{ m}$ . Similarly, since we know the width of the backyard is 18 m and the backyard is 3 rectangles across, the width of one rectangle is  $18 \div 3 = 6 \text{ m}$ .

Two adjacent sides of the middle rectangle and a diagonal form a right-angled triangle. Let  $d$  represent the length of the hypotenuse of the right-angled triangle with sides 8 m and 6 m. We know that  $d$  is also the length of the diameter of the pool. Using the Pythagorean Theorem,

$$\begin{aligned}d^2 &= 8^2 + 6^2 \\&= 64 + 36 \\&= 100 \\d &= \sqrt{100}, \text{ since } d > 0 \\d &= 10 \text{ m}\end{aligned}$$



Therefore, the diameter of the pool is 10 m, so it follows that the radius of the pool will be  $10 \div 2 = 5 \text{ m}$ .

The area of the pool can be calculated using the formula  $A = \pi r^2$  so  $A = \pi \times (5)^2 = 25\pi \approx 78.5 \text{ m}^2$ . Since  $78.5 < 86.4$ , the area of the pool is less than one-fifth of the total area of the backyard.

Therefore, their plan should be approved.



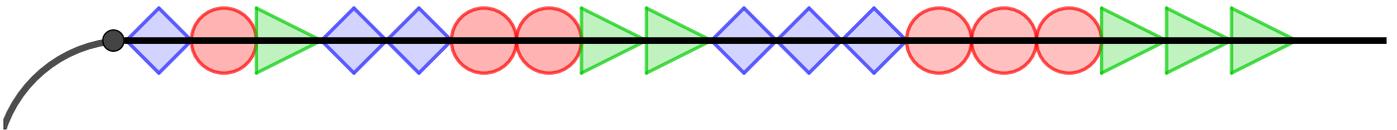
## Problem of the Week

### Problem C

### Shape On

A necklace is to be created that contains only square shapes, circular shapes, and triangular shapes. A total of 180 of these shapes will be strung on the necklace in the following sequence: 1 square, 1 circle, 1 triangle, 2 squares, 2 circles, 2 triangles, 3 squares, 3 circles, 3 triangles, with the number of each shape type increasing by one every time a new group of shapes is placed. The diagram illustrates how the first 18 shapes would be strung.

Once the necklace is completed, how many of each shape would the necklace contain?

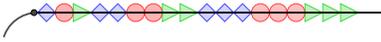




## Problem of the Week

### Problem C and Solution

#### Shape On



#### Problem

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Once the necklace is completed, how many of each shape would the necklace contain?

#### Solution

An equal number of square shapes, circular shapes and triangular shapes occur after

$$\begin{aligned}3(1) &= 3 \text{ shapes are placed,} \\3(1) + 3(2) &= 3 + 6 = 9 \text{ shapes are placed,} \\3(1) + 3(2) + 3(3) &= 3 + 6 + 9 = 18 \text{ shapes are placed, and so on.}\end{aligned}$$

The greatest total that can be placed with equal numbers of squares, circles and triangles is

$$\begin{aligned}3(1) + 3(2) + 3(3) + 3(4) + 3(5) + 3(6) + 3(7) + 3(8) + 3(9) + 3(10) \\&= 3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 + 30 \\&= 165\end{aligned}$$

At this point there are  $165 \div 3 = 55$  of each of the three shapes. We are at a point the last 30 shapes placed are 10 squares, 10 circles and 10 triangles, in that order.

The next group would have 11 of each shape if all of the shapes could be placed. However, there are only  $180 - 165 = 15$  shapes left to place. We are able to place 11 square shapes leaving 4 shapes left to place. At this point there are  $55 + 11 = 66$  square shapes on the necklace. The final 4 shapes would be circular. And at this point there would be 59 circular shapes on the necklace. No more triangular shapes can be added so the total number of triangular shapes remains at 55.

Therefore, the completed necklace will contain 66 square shapes, 59 circular shapes, and 55 triangular shapes.



## Problem of the Week

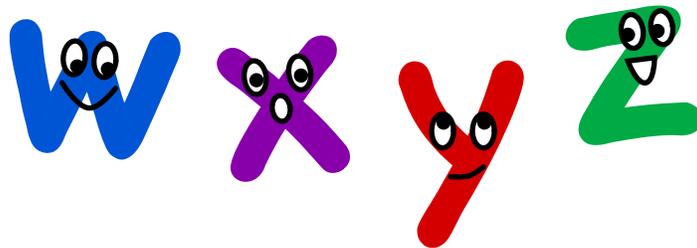
### Problem C

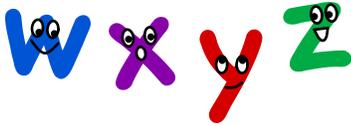
### Order Up!

The letters  $w$ ,  $x$ ,  $y$ , and  $z$  each represent a different positive integer greater than 3. If we know that

$$\frac{1}{w-3} = \frac{1}{x+1} = \frac{1}{y+2} = \frac{1}{z-2}$$

then write  $w$ ,  $x$ ,  $y$ , and  $z$  in order from the letter that represents the smallest integer to the letter that represents the largest integer.





## Problem of the Week

### Problem C and Solution

### Order Up!

#### Problem

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then write  $w$ ,  $x$ ,  $y$ , and  $z$  in order from the letter that represents the smallest integer to the letter that represents the largest integer.

#### Solution

##### Solution 1:

Since the fractions are all equal and they all have a numerator of 1, that means that their denominators must all be equal. So  $w - 3 = x + 1 = y + 2 = z - 2$ .

Now let's suppose that  $w = 10$ . Then  $w - 3 = 10 - 3 = 7$ .

So  $7 = x + 1 = y + 2 = z - 2$ . We can make the following conclusions.

- Since  $7 = x + 1$ , that means  $x = 7 - 1 = 6$ .
- Since  $7 = y + 2$ , that means  $y = 7 - 2 = 5$ .
- Since  $7 = z - 2$ , that means  $z = 7 + 2 = 9$ .

So when  $w = 10$ , we have  $x = 6$ ,  $y = 5$ , and  $z = 9$ . We can see that  $x$  is four less than  $w$ ,  $y$  is five less than  $w$ , and  $z$  is one less than  $w$ . So when we write these in order from smallest to largest, we get  $y$ ,  $x$ ,  $z$ ,  $w$ .

##### Solution 2:

As with Solution 1, we notice that since the fractions are all equal and they all have a numerator of 1, that means that their denominators must all be equal.

So  $w - 3 = x + 1 = y + 2 = z - 2$ . Let's add 3 to each expression.

$$\begin{array}{ccccccc} w - 3 & = & x + 1 & = & y + 2 & = & z - 2 \\ \downarrow +3 & & \downarrow +3 & & \downarrow +3 & & \downarrow +3 \\ w & = & x + 4 & = & y + 5 & = & z + 1 \end{array}$$

From this we can make the following conclusions.

- Since  $w = z + 1$ , that means  $w$  is 1 more than  $z$ , so  $w > z$ .
- Since  $z + 1 = x + 4$ , that means  $z$  is 3 more than  $x$ , so  $z > x$ .
- Since  $x + 4 = y + 5$ , that means  $x$  is 1 more than  $y$ , so  $x > y$ .

So when we write these in order from smallest to largest, we get  $y$ ,  $x$ ,  $z$ ,  $w$ .

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# Data Management (D)

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## Problem of the Week

### Problem C

### To the Next Level

Maddie Leet is participating in the first round of a math competition. She writes one contest a month in each of the first ten months of the school year. She can earn up to 100 points on each contest. On each of the first five contests she averaged 68 points. On the next three contests she averaged 80 points. In order to advance to the next round of the competition, she must obtain a minimum total of 750 points on the ten contests.

What is the minimum average Maddie requires on the final two contests in order to be able to advance to the next round of the competition?





## Problem of the Week

### Problem C and Solution

### To the Next Level

#### Problem

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#### Solution

To determine an average of some numbers, we add the numbers together and divide the sum by the number of numbers.

$$\text{Average} = \frac{\text{Sum of Numbers}}{\text{Number of Numbers}}$$

Therefore, to determine the sum of the numbers we multiply their average by the number of numbers.

$$\text{Sum of Numbers} = \text{Average} \times \text{Number of Numbers}$$

$$\text{Total Score for First 5 Contests} = 68 \times 5 = 340$$

$$\text{Total Score for Next 3 Contests} = 80 \times 3 = 240$$

$$\text{Total Score to Move On} = 750$$

$$\text{Total Score Needed for Last Two Contests} = 750 - 340 - 240 = 170$$

$$\text{Average Score for Final 2 Contests} = 170 \div 2 = 85$$

Maddie needs to average 85 points on her last two contests to have a total score of exactly 750 points. Therefore, the minimum average Maddie needs on the final two contests in order to advance to the next round of the math competition is 85 points. If she averages anything less than 85 points she will not move on. If she averages more than 85 points she will obtain a total score over 750 points and will move on to the next round.



## Problem of the Week

### Problem C

### An Odd Sum

Five cards are numbered 1 through 5. The back of each card is identical. The cards are mixed up and placed faced down. A person randomly flips over three cards and determines the sum of the three exposed cards. What is the probability that the sum is odd?





1

## Problem of the Week

### Problem C and Solution

#### An Odd Sum

5

#### Problem

Five cards are numbered 1 through 5. The back of each card is identical. The cards are mixed up and are placed faced down. A person randomly flips over three cards and determines the sum of the three exposed cards. What is the probability that the sum is odd?

#### Solution

In order to determine the probability, we must determine the number of ways to obtain a sum that is odd and divide it by the total number of possible selections of three cards.

We will count all of the possibilities by systematically listing the possible selections.

Select 1 and 2 and one higher number: 123, 124, 125; three possibilities.

Select 1 and 3 and one higher number: 134, 135; two possibilities.

Select 1 and 4 and one higher number: 145; one possibility.

Select 2 and 3 and one higher number: 234, 235; two possibilities.

Select 2 and 4 and one higher number: 245; one possibility.

Select 3 and 4 and one higher number: 345; one possibility.

By counting the outcomes from each case, there are  $3 + 2 + 1 + 2 + 1 + 1 = 10$  possible selections of three cards. We must now determine how many of these selections have an odd sum. We could take each of the possibilities, determine the sum and then count the number which produce an odd sum. However, we will present a different method which could be useful in other situations. The sum of three numbers is odd in two instances: there are three odd numbers or there is one odd number and two even numbers.

Selections with three odd numbers: 135.

Selections with one odd number and two even numbers: 124, 234 and 245.

The total number of selections where the sum is odd is  $1 + 3 = 4$ .

Therefore, the probability of selecting three cards with an odd sum is  $\frac{4}{10} = \frac{2}{5}$ .

#### Extending the problem:

If nine cards numbered 1 to 9 are mixed up and placed face down, then the probability of flipping over three cards that have an odd sum is  $\frac{10}{21}$ . Can you verify this?



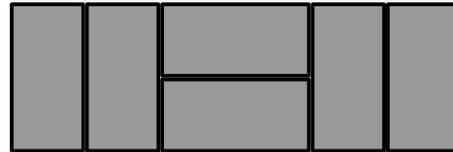
## Problem of the Week

### Problem C

#### A Path Using Math

A landscaper needs to fill a path measuring 2 feet by 6 feet with patio stones. The patio stones are each 1 foot by 2 feet, so the landscaper calculates that she will need 6 of them.

Before arranging the patio stones, the landscaper wants to look at all of her options. She cannot cut or overlap the stones, and they all must fit inside the path area without any gaps. Two possible arrangements of the stones are shown below. How many different arrangements are there in total?





## Problem of the Week

### Problem C and Solution

#### A Path Using Math



### Problem

A landscaper needs to fill a path measuring 2 feet by 6 feet with patio stones. The patio stones are each 1 foot by 2 feet, so the landscaper calculates that she will need 6 of them. Before arranging the patio stones, the landscaper wants to look at all of her options. She cannot cut or overlap the stones, and they all must fit inside the path area without any gaps. Two possible arrangements of the stones are shown. How many different arrangements are there in total?

### Solution

Let's consider the ways that the patio stones can be arranged. We will imagine we are looking at the path from the side, just like in the images shown in the question. First, notice that there must always be an even number of patio stones that have a horizontal orientation, because they must be placed in pairs.

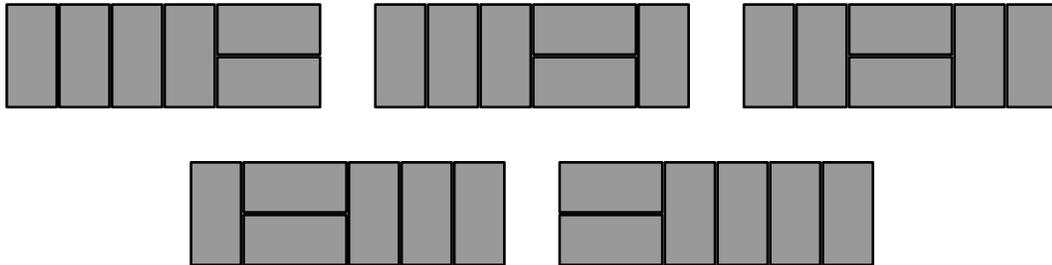
- All patio stones are vertical (and none are horizontal)

This can be done in only one way.



- Four patio stones are vertical and two are horizontal

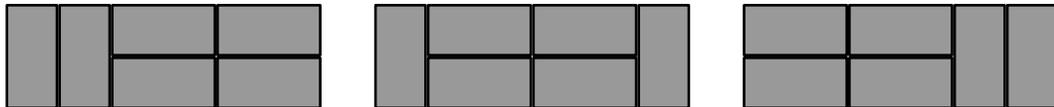
There could be 0, 1, 2, 3 or 4 vertical stones to the right of the horizontal stones. So there are 5 ways that four patio stones are vertical and two are horizontal.



- Two patio stones are vertical and four are horizontal

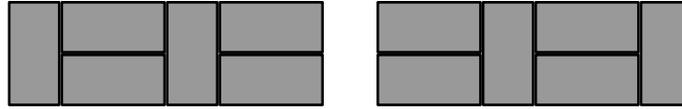
We need to consider sub cases:

- Case 1: There are no vertical stones between the horizontal stones.  
There could be 0, 1 or 2 vertical stones to the right of the horizontal stones. So there are 3 ways that two patio stones are vertical and four are horizontal when there are no vertical stones between the horizontal stones.





- Case 2: There is one vertical stone between the horizontal stones.  
There could be 0 or 1 vertical stones to the right of the rightmost horizontal stones.  
So there are 2 ways that two patio stones are vertical and four are horizontal when there is one vertical stone between the horizontal stones.

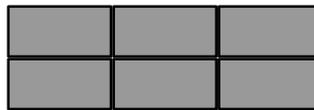


- Case 3: There are two vertical stones between the horizontal stones.  
There cannot be any vertical stones to the right of the rightmost horizontal stones, since all the vertical stones are between the horizontal stones. So there is 1 way that two patio stones are vertical and four are horizontal when there are two vertical stones between the horizontal stones.



- All patio stones are horizontal

This can be done in only one way.



Therefore, there are a total of  $1 + 5 + (3 + 2 + 1) + 1 = 13$  different arrangements of the patio stones.



## Problem of the Week

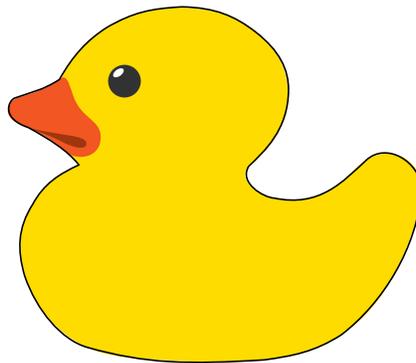
### Problem C

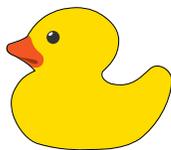
### Lucky Ducks

For a school fundraiser, Percy bought a big box of rubber ducks and wrote prize amounts, in dollars, on the bottom of each rubber duck. The prize amounts were \$5, \$10, \$20, \$50, and \$100. The number of ducks with each prize amount varied so that the total value for all the ducks with each prize amount was always \$500.

Percy then put all the rubber ducks in his school's swimming pool. At the fundraiser, participants used a long net to catch a duck from the pool. They won the amount written on the bottom of the rubber duck, and then they threw the duck back into the pool before the next person's turn. Since the ducks were returned to the pool after they were caught, the chances of winning any particular amount remained the same.

Amir paid \$15 to play the game once. If we assume he randomly selected a duck, what is the probability that Amir won more money than he paid to play the game?





## Problem of the Week

### Problem C and Solution

#### Lucky Ducks

#### Problem

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Amir paid \$15 to play the game once. If we assume he randomly selected a duck, what is the probability that Amir won more money than he paid to play the game?

#### Solution

In order to determine the probability, we need to know the total number of ducks with each prize amount written on the bottom. These are calculated in the table below.

Prize Amount	Total Value of All Prizes with this Amount	Number of Ducks with this Prize Amount
\$5	\$500	$\$500 \div \$5 = 100$
\$10	\$500	$\$500 \div \$10 = 50$
\$20	\$500	$\$500 \div \$20 = 25$
\$50	\$500	$\$500 \div \$50 = 10$
\$100	\$500	$\$500 \div \$100 = 5$

The total number of ducks is  $100 + 50 + 25 + 10 + 5 = 190$ .

The total number of ducks containing a prize amount greater than \$15 is  $25 + 10 + 5 = 40$ .

Therefore, the probability that Amir won more than \$15 is:

$$\frac{\text{number of ducks containing a prize amount greater than \$15}}{\text{total number of ducks}} = \frac{40}{190} = \frac{4}{19}$$

That is, the probability that Amir won more than \$15 is  $\frac{4}{19}$ .

So Amir has approximately a 21% chance of winning more than he paid.



## Problem of the Week

### Problem C

### CADET

We can take any word and rearrange all the letters to get another “word”. These new “words” may be nonsensical. For example, you can rearrange the letters in *MATH* to get *MTHA*. Geordie wants to rearrange all the letters in the word *CADET*. However, he uses the following rules:

- the letters *A* and *D* must be beside each other, and
- the letters *E* and *T* must be beside each other.

How many different arrangements of the word *CADET* can Geordie make if he follows these rules?



NOTE: *ADCET* and *ADTEC* are acceptable words, while *ADTCE* is not.



## Problem of the Week

### Problem C and Solution

### CADET

#### Problem

We can take any word and rearrange all the letters to get another “word”. These new “words” may be nonsensical. For example, you can rearrange the letters in *MATH* to get *MTHA*. Geordie wants to rearrange all the letters in the word *CADET*. However, he uses the following rules:

- the letters *A* and *D* must be beside each other, and
- the letters *E* and *T* must be beside each other.

How many different arrangements of the word *CADET* can Geordie make if he follows these rules?

#### Solution

We will look at a systematic way of counting the arrangements by first looking at a simpler example.

Let’s start with a three letter word. How many different ways can we arrange the letters of the word *SPY*?

If we list all the arrangements, we get 6 arrangements.

They are: *SPY, SYP, PYS, PSY, YPS, YSP*.

There is another way to count these 6 cases without listing them all out:

If we consider the first letter, there are 3 possibilities. For each of these possibilities, there are 2 remaining options for the second letter. Finally, once the first and second letters are set, there is only one possibility left for the last letter. To get the number of possible arrangements, we multiply  $\underline{3} \times \underline{2} \times \underline{1} = 6$ .

Let’s look at our problem now.

If we consider *A* and *D* as the single “letter” *AD*, and *E* and *T* as the single “letter” *ET*, we now have only the three “letters” *C, AD, and ET*.

As we saw above, there are  $\underline{3} \times \underline{2} \times \underline{1} = 6$  ways to arrange the three “letters”. These arrangements are:

*CADET, CETAD, ADCET, ADETC, ETCAD, ETADC*.

(We will refer to these as the original six.)

However, note that the question says *A* and *D* must be beside each other. This means they could appear as *AD* or *DA*. Similarly, *E* and *T* could appear as *ET* or *TE*.



Let's take a look at the first word, *CADET*.

We could switch *AD* to *DA*. This means *CADET* becomes *CDAET*, which is a valid arrangement.

We could also switch *ET* to *TE*. This means *CADET* becomes *CADTE*, which is a valid arrangement.

We could also switch both *AD* to *DA* and *ET* to *TE*. This means *CADET* becomes *CDATE*, which is a valid arrangement.

We can do these changes for each of the original six. We list these arrangements in the table below.

One of the Original Six	Switch <i>AD</i>	Switch <i>ET</i>	Switch both <i>AD</i> and <i>ET</i>
<i>CADET</i>	<i>CDAET</i>	<i>CADTE</i>	<i>CDATE</i>
<i>CETAD</i>	<i>CETDA</i>	<i>CTEAD</i>	<i>CTEDA</i>
<i>ADCET</i>	<i>DACET</i>	<i>ADCTE</i>	<i>DACTE</i>
<i>ADETC</i>	<i>DAETC</i>	<i>ADTEC</i>	<i>DATEC</i>
<i>ETCAD</i>	<i>ETCDA</i>	<i>TECAD</i>	<i>TECDA</i>
<i>ETADC</i>	<i>ETDAC</i>	<i>TEADC</i>	<i>TEDAC</i>

Therefore, there are 24 possible arrangements that Geordie can make when following the given rules.

NOTE: There is another way to count the number of arrangements. There are 6 ways to arrange the 3 "letters". There are 2 ways to arrange *AD* and there are 2 ways to arrange *ET*. To determine the total number of arrangements, we multiply  $6 \times 2 \times 2$ . This gives us 24 arrangements.

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# Computational Thinking (C)

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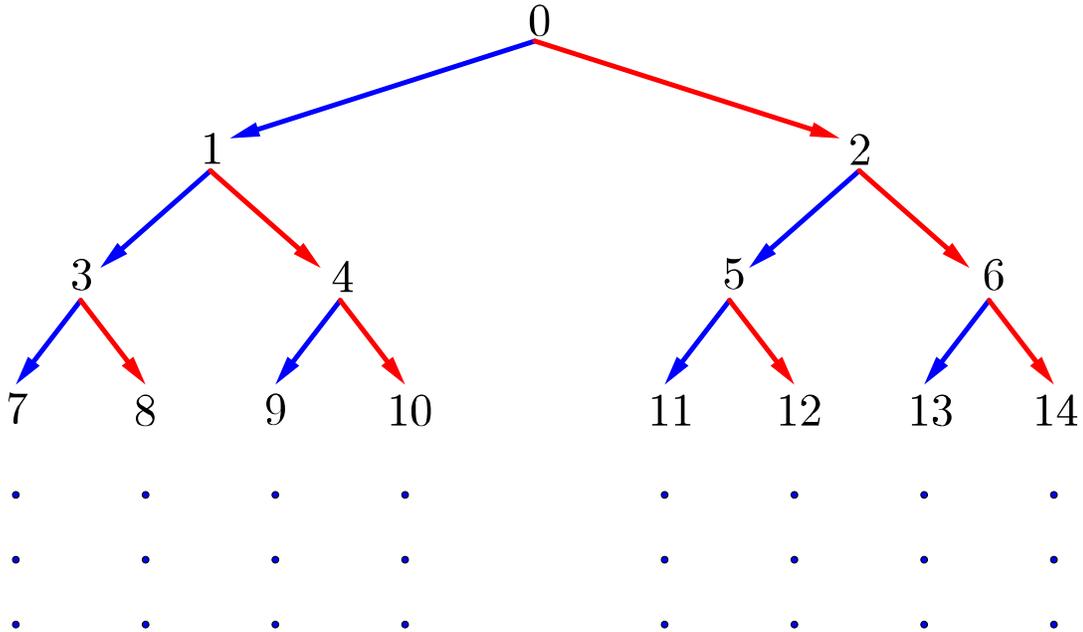


# Problem of the Week

## Problem C

### Branching Out

Consider the following number tree.



In this number tree, the integers greater than or equal to 0 are written out in increasing order, with the top row containing one integer and every row after containing twice as many integers as the row above it.

If row 1 contains the integer 0, what is the fifth number in row 10?



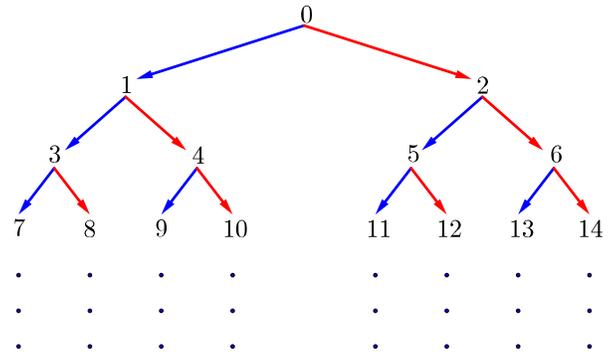
# Problem of the Week

## Problem C and Solution

### Branching Out

#### Problem

Consider the number tree shown to the right. In this number tree, the integers greater than or equal to 0 are written out in increasing order, with the top row containing one integer and every row after containing twice as many integers as the row above it. If row 1 contains the integer 0, what is the fifth number in row 10?



#### Solution

##### Solution 1

One approach to solving the problem would be to write out the first 10 rows of the chart and read off the fifth number in row 10. You would discover that the fifth integer in row 10 is 515. This solution may “work” in this example but it is certainly not ideal. It would not be practical if you were asked for the seventh integer in row 50.

##### Observations

There are many patterns in the tree. The solutions provided below will look at some of the different patterns which can be used to solve the problem.

##### Solution 2

Row 1 contains 1 integer, row 2 contains 2 integers, row 3 contains 4 integers, and row 4 contains 8 integers. Each new row in the tree has twice as many integers as the previous row.

Using this, we could find the number of integers in the first 9 rows of the chart. There are

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 = 511$$

integers in the first 9 rows of the chart. There are 511 integers in the 9 rows, but the first integer in the chart is 0. Therefore, it follows that the last integer in the 9th row is 1 less than the number of integers in 9 rows. That is, the last integer in row 9 is 510. Therefore, the first integer in row 10 is 511. We can easily count to the fifth spot to obtain 515, as above.

**Solution 3**

This solution is similar to Solution 2, but only looks at the rightmost number in each row.

To get from the top number to the rightmost number in row 2, add 2. To get from the rightmost number in row 2 to the rightmost number in row 3, add 4. To get from the rightmost number in row 3 to the rightmost number in row 4, add 8. The numbers that are added correspond to the number of integers in the next row. We will find the rightmost number in row 9 and then add 1 to get the leftmost number in row 10. The rightmost number in row 9 is:

$$0 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 = 510$$

This tells us that the leftmost number in row 10 is 511. Therefore, the fifth number in row 10 is 515.

**Solution 4**

Since each row after the first has twice as many integers as the row above, there is some connection to powers of 2 in the problem. The following table shows the row number, the rightmost number in that row, the power of 2 with the row number as the exponent, and the connection between this power and the last number in the row.

Row Number	Rightmost Number in Row	Power of 2	Connection
1	0	$2^1 = 2$	$2^1 - 2 = 2 - 2 = 0$
2	2	$2^2 = 4$	$2^2 - 2 = 4 - 2 = 2$
3	6	$2^3 = 8$	$2^3 - 2 = 8 - 2 = 6$
4	14	$2^4 = 16$	$2^4 - 2 = 16 - 2 = 14$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n$	??	$2^n$	$2^n - 2 = ??$

It would appear that the last number in row 5 should be  $2^5 - 2 = 32 - 2 = 30$ . We could write out the fifth row to confirm that this is correct.

It would also appear that the last number in row  $n$  should be  $2^n - 2$ . By recognizing the pattern, we predict that the last number in row 9 should be  $2^9 - 2 = 512 - 2 = 510$ . We know from earlier solutions that this is correct.

Therefore, the first number in row 10 is 511 and the fifth number in row 10 is 515.

It should be noted that this relationship works for all of the rows we have sampled, but we have not proven that it is true in general. You will have to wait for some higher mathematics to be able to prove that this is true in general.

The pattern used in this solution is not an obvious one, but by discovering it the solution became fairly straightforward. In fact, if we accept the result as true, then we can quickly determine the value of the rightmost integer in any row with a simple calculation.



## Problem of the Week

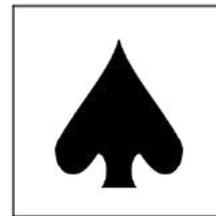
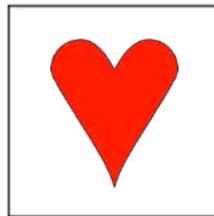
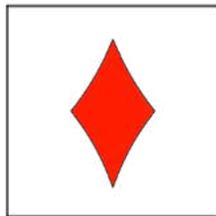
### Problem C

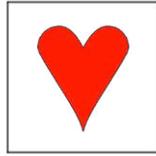
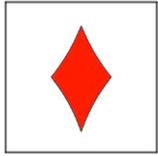
#### Order Three Cards

Three playing cards are placed in a row, from left to right. Each card is of a different suit. One card is a diamond ( $\diamond$ ), one card is a heart ( $\heartsuit$ ), and one card is a spade ( $\spadesuit$ ). The number on each card is also different. One card is a 3, one card is a 7, and one card is a 9.

Using the following clues, determine the exact order of the cards, from left to right, including the suit and number.

1. The diamond is somewhere to the right of the spade.
2. The 7 is somewhere to the left of the spade.
3. The 9 is somewhere to the right of the 3.

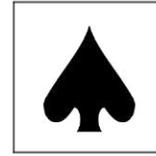




## Problem of the Week

### Problem C and Solution

#### Order Three Cards



### Problem

Three playing cards are placed in a row, from left to right. Each card is of a different suit. One card is a diamond ( $\diamond$ ), one card is a heart ( $\heartsuit$ ), and one card is a spade ( $\spadesuit$ ). The number on each card is also different. One card is a 3, one card is a 7, and one card is a 9.

Using the following clues, determine the exact order of the cards, from left to right, including the suit and number.

1. The diamond is somewhere to the right of the spade.
2. The 7 is somewhere to the left of the spade.
3. The 9 is somewhere to the right of the 3.

### Solution

There are six ways to order the suits:

$(\diamond, \heartsuit, \spadesuit)$ ,  $(\diamond, \spadesuit, \heartsuit)$ ,  $(\heartsuit, \diamond, \spadesuit)$ ,  $(\heartsuit, \spadesuit, \diamond)$ ,  $(\spadesuit, \diamond, \heartsuit)$ , and  $(\spadesuit, \heartsuit, \diamond)$ .

The first clue tells us that the diamond is somewhere to the right of the spade. We can eliminate the first three orders from the above list since the diamond is to the left of the spade in each case. There are now only three possible ways to order the suits:

$(\heartsuit, \spadesuit, \diamond)$ ,  $(\spadesuit, \diamond, \heartsuit)$ , and  $(\spadesuit, \heartsuit, \diamond)$ .

The second clue says that the 7 is somewhere to the left of the spade. This means that the spade cannot be the leftmost card. This eliminates the last two possibilities from the above list. The only possibility is  $(\heartsuit, \spadesuit, \diamond)$  and the 7 must be the 7 of hearts giving us  $(7\heartsuit, \spadesuit, \diamond)$ .

The third clue tells us that the 9 is somewhere to the right of the 3. With only two spots to decide, we can conclude that the 9 must be in the rightmost (third) spot and the last two cards are the 3 of spades and the 9 of diamonds. This gives us  $(7\heartsuit, 3\spadesuit, 9\diamond)$ .

Therefore, the cards are placed in the following order from left to right: 7 of hearts, then 3 of spades, and then 9 of diamonds.



## Problem of the Week Problem C Step by Step

Debit and credit cards contain account numbers which consist of many digits. When purchasing items online, you are often asked to type in your account number. Because there are so many digits, it is easy to type the number incorrectly. The last digit of the number is a specially generated check digit which can be used to quickly verify the validity of the number. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid.

The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.

### Example 1:

Number: 135792

Reversal: 297531

$$A = 2 + 7 + 3 \\ = 12$$

$$2 \times 9 = 18 \\ 2 \times 5 = 10 \\ 2 \times 1 = 2$$

$$B = (1 + 8) + (1 + 0) + 2 \\ = 9 + 1 + 2 \\ = 12$$

$$C = 12 + 12 = 24$$

$C$  does not end in zero.  
The number is not valid.

### Example 2:

Number: 1357987

Reversal: 7897531

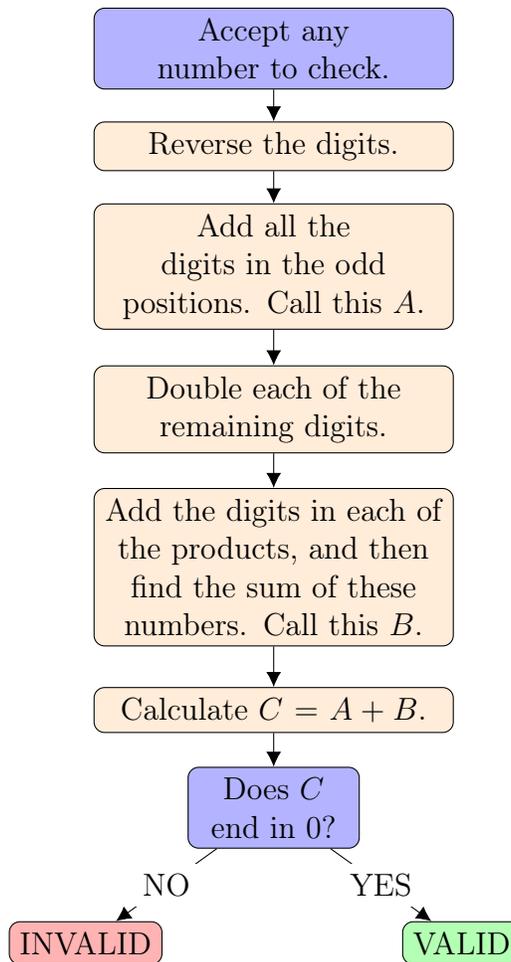
$$A = 7 + 9 + 5 + 1 \\ = 22$$

$$2 \times 8 = 16 \\ 2 \times 7 = 14 \\ 2 \times 3 = 6$$

$$B = (1 + 6) + (1 + 4) + 6 \\ = 7 + 5 + 6 \\ = 18$$

$$C = 22 + 18 = 40$$

$C$  ends in zero.  
The number is valid.



The number 1953  $R$ 8 $T$ 9 467 is a valid card number when verified by the Luhn Algorithm.  $R$  and  $T$  are each single digits of the number such that  $R$  is less than  $T$ .

Determine all possible values of  $R$  and  $T$ .



## Problem of the Week

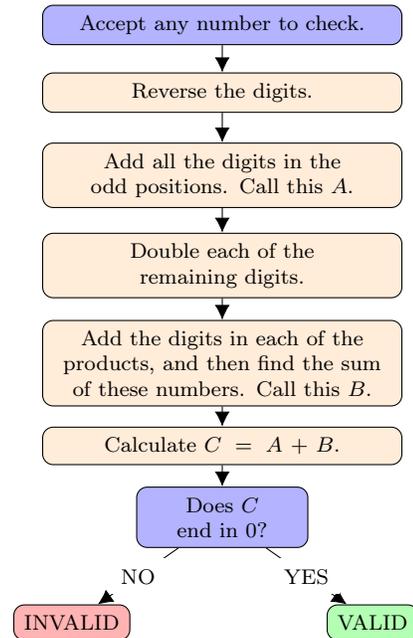
### Problem C and Solution

#### Step by Step

### Problem

Debit and credit cards contain account numbers which consist of many digits. When purchasing items online, you are often asked to type in your account number. Because there are so many digits, it is easy to type the number incorrectly. The last digit of the number is a specially generated check digit which can be used to quickly verify the validity of the number. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid.

The steps performed in the Luhn Algorithm are outlined in the flowchart to the right.



The number 1953 *RST*9 467 is a valid card number when verified by the Luhn Algorithm. *R* and *T* are each single digits of the number such that *R* is less than *T*. Determine all possible values of *R* and *T*.

### Solution

When the digits of the card number are reversed the resulting number is 764 9*T*8*R* 3591. The sum of the digits in the odd positions is  $A = 7 + 4 + T + R + 5 + 1 = 17 + R + T$ .

When the digits in the remaining positions are doubled, the following products are obtained:

$$6 \times 2 = 12; 9 \times 2 = 18; 8 \times 2 = 16; 3 \times 2 = 6; \text{ and } 9 \times 2 = 18$$

When the digit sums from each of the products are added, the sum is:

$$B = (1 + 2) + (1 + 8) + (1 + 6) + 6 + (1 + 8) = 3 + 9 + 7 + 6 + 9 = 34$$

Since  $C = A + B$ , we have  $C = 17 + R + T + 34 = 51 + R + T$ .

For the card to be valid, the units digit of *C* must be zero. The closest number greater than 51 ending in a zero is 60. It follows that  $R + T = 60 - 51 = 9$ . We want values of *R* and *T* that sum to 9 with *R* less than *T*. The only possible combinations are *R* = 0 and *T* = 9, *R* = 1 and *T* = 8, *R* = 2 and *T* = 7, *R* = 3 and *T* = 6, and *R* = 4 and *T* = 5.

The next closest number greater than 51 ending in a zero is 70. It follows that  $R + T = 70 - 51 = 19$ . But the largest value possible for *T* is 9 and the largest value for *R* would then be 8. It follows that the largest possible value for  $R + T = 8 + 9 = 17$ . This means that there are no values of *R* and *T* so that *C* could equal 70.

Therefore, there are only 5 possible pairs of digits for *R* and *T* so that 1953 *RST*9 467 is a valid card number. The possibilities are *R* = 0 and *T* = 9, *R* = 1 and *T* = 8, *R* = 2 and *T* = 7, *R* = 3 and *T* = 6, and *R* = 4 and *T* = 5.

The card numbers are 1953 0899 467, 1953 1889 467, 1953 2879 467, 1953 3869 467, and 1953 4859 467, which are indeed valid by the Luhn Algorithm.



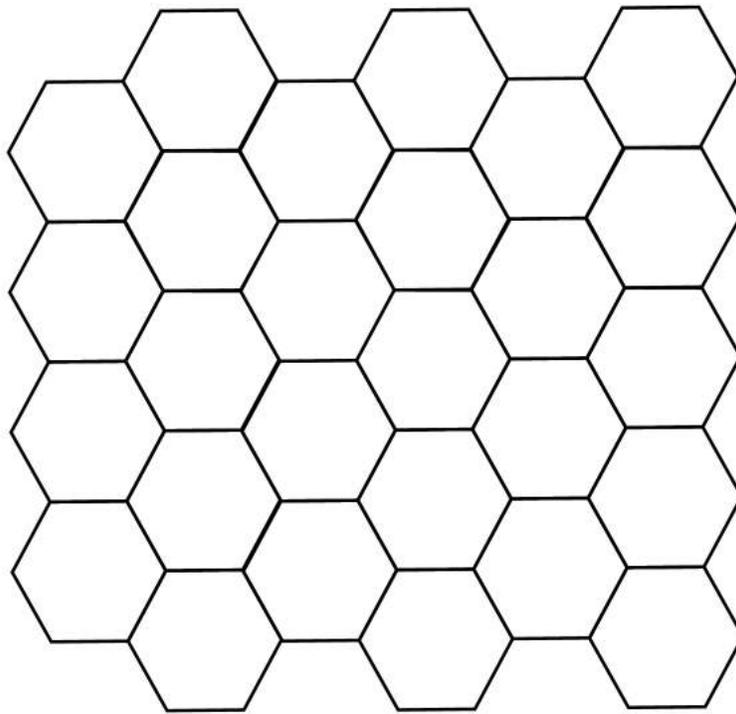
## Problem of the Week

### Problem C

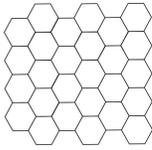
#### Colour by Numbers

Jessica is going to colour the hexagons in the tiling below. Each hexagon will be coloured with a single colour. If two hexagons share a side, then they will be coloured with a different colour.

What is the fewest number of colours that Jessica can use to colour all the hexagons?



**Not printing this page?** You can colour the hexagons on our [interactive worksheet](#).



## Problem of the Week

### Problem C and Solution

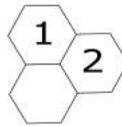
#### Colour by Numbers

#### Problem

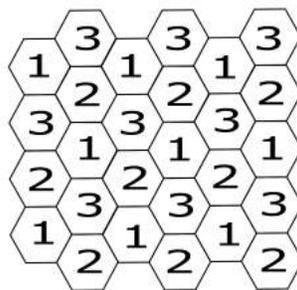
Jessica is going to colour the hexagons in the tiling below. Each hexagon will be coloured with a single colour. If two hexagons share a side, then they will be coloured with a different colour. What is the fewest number of colours that Jessica can use to colour all the hexagons?

#### Solution

We first determine if it is possible for her to use only two colours (using just one colour is not possible). We will use the numbers 1, 2, 3 to represent distinct (different) colours. We begin by choosing any group of three hexagons in which each pair of hexagons share a side, as shown. We colour two of the hexagons with colours 1 and 2 (since they share a side). Each of these two coloured hexagons shares a side with the third hexagon, which therefore cannot be coloured 1 or 2.



Thus, the minimum number of colours that Jessica can use is at least three. Next, we determine if the entire tiling can be coloured using only three colours. Here is one possible colouring of the hexagons that uses only three colours.



While other colourings of the hexagons are possible, we can see that it is possible for Jessica to use only three colours and ensure that no two hexagons that share a side are the same colour.

Therefore, the fewest number of colours that Jessica can use to colour all the hexagons is three.

There are many nice patterns of the colours in this tiling. Can you find a different colouring of the tiles that uses only three colours?