The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Problem of the Week Problems and Solutions 2020-2021

Problem E (Grade 11/12)

Themes

Number Sense (N)
Geometry (G)
Algebra (A)
Data Management (D)
Computational Thinking (C)

(Click on a theme name above to jump to that section)

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*The problems in this booklet are organized into themes.
A problem often appears in multiple themes.

Number Sense (N)

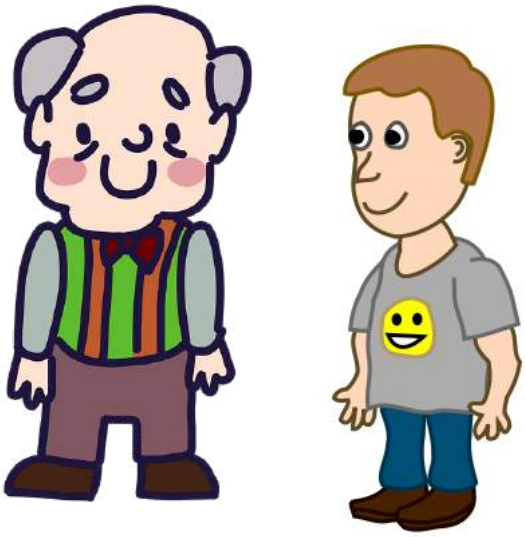
TAKE ME TO THE COVER

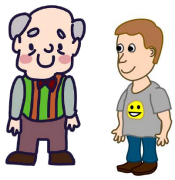
Problem of the Week

Problem E

More of Grandpa's Math

Arden's Grandpa tries to challenge him in his mathematics studies by creating problems for him to solve. For one problem, Grandpa said, "I am thinking of some positive integer, let's call it k, which I then multiply by the number 1357 and I end up with a product whose final four digits are 9502. Find the smallest value of k that will make this work."





Problem of the Week
Problem E and Solution
More of Grandpa's Math

Problem

Arden's Grandpa tries to challenge him in his mathematics studies by creating problems for him to solve. For one problem, Grandpa said, "I am thinking of some positive integer, let's call it k, which I then multiply by the number 1357 and I end up with a product whose final four digits are 9502. Find the smallest value of k that will make this work."

Solution

To begin with, we will show that k has four digits or less. A number with five digits, $pqrst$ for example, can be written $p \times 10^4 + qrst = p0000 + qrst$. The digit p in the multiplier cannot affect the final four digits in the product. Therefore, the minimum k is a number with four or fewer digits.

Let the multiplier be $abcd$ such that $1357 \times abcd$ is a number whose last four digits are 9502.

Then multiplying 7, the units digit of 1357, by d, the units digit in $abcd$, produces a number ending in 2. The only possible value for d is 6 since $7 \times 6 = 42$. (Note the possible last digits when 7 multiplies a single digit number: $7 \times 0 = 0,\ 7 \times 1 = 7,\ 7 \times 2 = 14,\ 7 \times 3 = 21,\ 7 \times 4 = 28,$ $7 \times 5 = 35,\ 7 \times 6 = 42,\ 7 \times 7 = 49,\ 7 \times 8 = 56,\ 7 \times 9 = 63$.) Therefore the multiplier is $abc6$.

The second last digit in the product 9502 is 0. This digit is produced by multiplying 57 from 1357 with $c6$ from $abc6$.

$$\begin{array}{rrr} & 5 & 7 \\ \times & c & 6 \\ \hline 3 & 4 & 2 \\ . & 7c & 0 \\ \hline . & 0 & 2 \end{array}$$

So $4 + 7c$ is a number that ends in 0. The only possible value for c is 8. (Refer back to the product list given above.) Therefore the multiplier is a number of the form $ab86$. Is $k = 86$? The product, $1357 \times 86 = 116\,702$, does not end in 9502. So k is at least a three digit number.

The third last digit in the product 9502 is 5. This digit is produced by multiplying 357 from 1357 with $b86$ from $ab86$.

$$\begin{array}{rrrrr} & & 3 & 5 & 7 \\ \times & & b & 8 & 6 \\ \hline & 2 & 1 & 4 & 2 \\ 2 & 8 & 5 & 6 & 0 \\ . & . & 7b & 0 & 0 \\ \hline . & . & 5 & 0 & 2 \end{array}$$

So $1 + (1 + 5 + 7b)$ is a number that ends in 5 (the extra 1 comes from the "carry" from the column to the right) and it follows that $7b$ is a number that ends in 8. The only possible value for b is 4. (Refer back to the product list given above.) Therefore the multiplier is a number of the form $a486$. Is $k = 486$? The product $1357 \times 486 = 659\,502$, which does end in 9502. Therefore, the smallest value of k is 486.

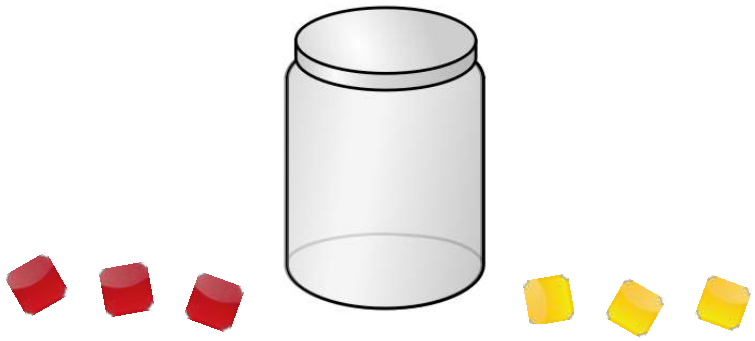
Problem of the Week

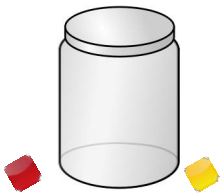
Problem E

Candies Anyone?

A jar contains only small red and small yellow candies. Another 30 red candies are added to the candies already in the jar so that one-third of the total number of candies in the jar are red candies. At this point, 30 yellow candies are added to the jar, and now three-tenths of the total number of candies in the jar are red candies.

What fraction of the number of the candies originally in the jar were red candies?





Problem of the Week

Problem E and Solution

Candies Anyone?

Problem

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Solution

Let r represent the number of red candies originally in the jar.
Let y represent the number of yellow candies originally in the jar.
Then $r + y$ represents the total number of candies originally in the jar.

After adding 30 red candies to the candies already in the jar, there are $(r + 30)$ red candies in the jar and a total of $(r + y + 30)$ candies in the jar. Now one-third of the candies in the jar are red candies, so

$$\begin{aligned} \frac{r+30}{r+y+30} &= \frac{1}{3} \\ 3(r+30) &= 1(r+y+30) \\ 3r+90 &= r+y+30 \\ 2r+60 &= y \qquad (1) \end{aligned}$$

After adding 30 yellow candies to the candies in the jar, there are $(r + 30)$ red candies in the jar and a total of $(r + y + 60)$ candies in the jar. Now three-tenths of the candies in the jar are red candies, so

$$\begin{aligned} \frac{r+30}{r+y+60} &= \frac{3}{10} \\ 10(r+30) &= 3(r+y+60) \\ 10r+300 &= 3r+3y+180 \\ 7r-3y &= -120 \qquad (2) \end{aligned}$$

Substituting (1) into (2),

$$\begin{aligned} 7r-3(2r+60) &= -120 \\ 7r-6r-180 &= -120 \\ r &= 60 \end{aligned}$$

Substituting $r = 60$ into (1), we get $y = 180$.

Therefore, there were originally 60 red and 180 yellow candies in the jar, and $\frac{60}{60+180} = \frac{60}{240} = \frac{1}{4}$ of the candies originally in the jar were red.

Problem of the Week

Problem E

All the Children

My family has four children, each with a different age. The product of their ages is 17 280. The sum of the ages of the three oldest children is 40 and the sum of the ages of the three youngest children is 32.

Determine all possibilities for the ages of the four children.

Problem of the Week
Problem E and Solution
All the Children

Problem

My family has four children, each with a different age. The product of their ages is 17 280. The sum of the ages of the three oldest children is 40 and the sum of the ages of the three youngest children is 32. Determine all possibilities for the ages of the four children.

Solution

Let the ages of the children from youngest to oldest be $a,\ b,\ c,\ d$.

Since the ages of the three oldest children sum to 40, $b + c + d = 40$. (1)
Since the ages of the three youngest children sum to 32, $a + b + c = 32$. (2)

Subtracting (2) from (1), we obtain $d - a = 8$. This means that the difference between the age of the oldest child and the age of the youngest child is 8.

Now we can factor 17 280.
$17\,280 = 2^7 \times 3^3 \times 5 = (2^2 \times 3) \times (2^2 \times 3) \times (2^2 \times 3) \times (2 \times 5) = 12 \times 12 \times 12 \times 10$. Since the children all have different ages, we can use this statement to help us figure out the possible ages of the oldest and youngest children.

Is it possible that the oldest child is 12? If the oldest child is 12, then the youngest child would be $12 - 8 = 4$. The largest product that could be generated using two more different ages between 4 and 12 would be $12 \times 11 \times 10 \times 4 = 5280$. Since this is less than 17 280, it follows that the oldest child must be older than 12.

Is it possible that the youngest child is 10? If the youngest child is 10, then the oldest child would be $10 + 8 = 18$. The smallest product that could be generated using two more different ages between 10 and 18 would be $10 \times 11 \times 12 \times 18 = 23\,760$. Since this is greater than 17 280, it follows that the youngest child must be younger than 10.

There are now a limited number of possibilities to consider for the ages of the youngest and oldest children. The possibilities are $(5, 13)$, $(6, 14)$, $(7, 15)$, $(8, 16)$, and $(9, 17)$. No other combinations are possible since the oldest child must be older than 12 and the youngest child must be younger than 10.

The prime numbers 7, 13, and 17 are not factors of 17 280. Therefore we can eliminate the possibilities where an age is one of 7, 13, or 17, leaving $(6, 14)$ and $(8, 16)$. Since 14 is a multiple of 7, we can conclude that it is also not a factor of 17 280. Thus, we can eliminate $(6, 14)$. Now there is only one possibility left to consider, namely $(8, 16)$.

Now $17\,280 = 8 \times 16 \times 3^3 \times 5$. Using the remaining factors 3^3 and 5, we need to create two numbers between 8 and 16. The only possibilities are $3^2 = 9$ and $3 \times 5 = 15$.

Therefore, the only possibility is that the ages of the children are 8, 9, 15, and 16. It is easy to verify that this is a valid solution.

Problem of the Week

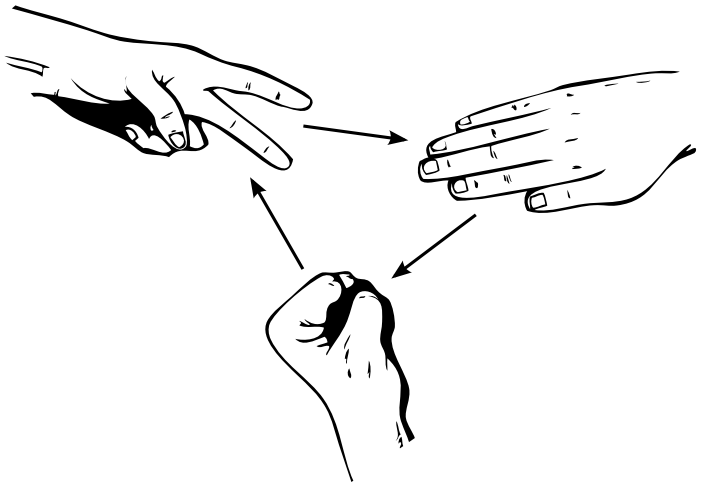
Problem E

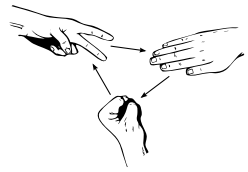
Rock Out

Sameer likes playing rock paper scissors so much that he started a club at his school. At every club meeting, each person plays rock paper scissors against each of the other people at the meeting the same number of times.

There were 40% more people at the second club meeting than the first. The total number of rock paper scissors matches played at the second meeting was twice the number of matches played at the first meeting. The number of matches each person plays against each of the other people at the meeting stays the same from meeting to meeting.

How many people were at the first meeting?





Problem of the Week
Problem E and Solution
Rock Out

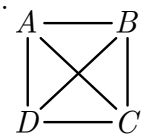
Problem

Sameer likes playing rock paper scissors so much that he started a club at his school. At every club meeting, each person plays rock paper scissors against each of the other people at the meeting the same number of times. There were 40% more people at the second club meeting than the first. The total number of rock paper scissors matches played at the second meeting was twice the number of matches played at the first meeting. The number of matches each person plays against each of the other people at the meeting stays the same from meeting to meeting. How many people were at the first meeting?

Solution

Let p represent the number of people at the first meeting. Then the number of people at the second meeting is $1.4p$. Note that both p and $1.4p$ must be positive integers.

We need to first establish how many games are played. Suppose there were 4 people ($A, B, C,$ and D), and each person played against every other person once, as shown to the right.



We can see there would be 6 matches played ($AB, AC, AD, BC, BD,$ and CD). This is easy to see in a small diagram, but as the number of people and matches increase, the diagrams become hard to read, so we need to find a general solution.

Often, when counting something like this we "double-count" by mistake. We think that because there are 4 people and each person plays the 3 other people, that means there are $4 \times 3 = 12$ matches in total. However, we have counted each match twice, so we need to divide the result by 2. So if there were 4 people, with each person playing every other person once, there would be $\frac{4\times 3}{2} = 6$ matches played in total.

In general, if there are p people and each person plays against every other person once, there would be $\frac{p(p-1)}{2}$ matches played in total. If each person plays against every other person n times, the total number of matches would need to be multiplied by n, to obtain $n\left(\frac{p(p-1)}{2}\right)$. This represents the total number of matches at the first meeting. Now in the second meeting, the number of people is $1.4p$, so the total number of matches in the second meeting would be $n\left(\frac{1.4p(1.4p-1)}{2}\right)$.

We know that the number of matches played at the second meeting is twice the number of matches played at the first meeting. So,

$$n\left(\frac{1.4p(1.4p-1)}{2}\right) = 2\left[n\left(\frac{p(p-1)}{2}\right)\right]$$

Dividing both sides by $\frac{n}{2}$, $n \neq 0$, this simplifies to

$$1.4p(1.4p-1) = 2p(p-1)$$

Dividing both sides by p, $p \neq 0$, this simplifies to

$$\begin{aligned} 1.4(1.4p-1) &= 2(p-1) \\ 1.96p - 1.4 &= 2p - 2 \\ 0.6 &= 0.04p \\ 15 &= p \end{aligned}$$

Therefore, there were 15 people at the first meeting.

Problem of the Week
Problem E
Is It Valid?

Debit and credit cards contain account numbers which consist of many digits. When purchasing items online, you are often asked to type in your account number. Because there are so many digits, it is easy to type the number incorrectly. The last digit of the number is a specially generated check digit which can be used to quickly verify the validity of the number. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid.

The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.

Example 1:

Number: 135792

Reversal: 297531

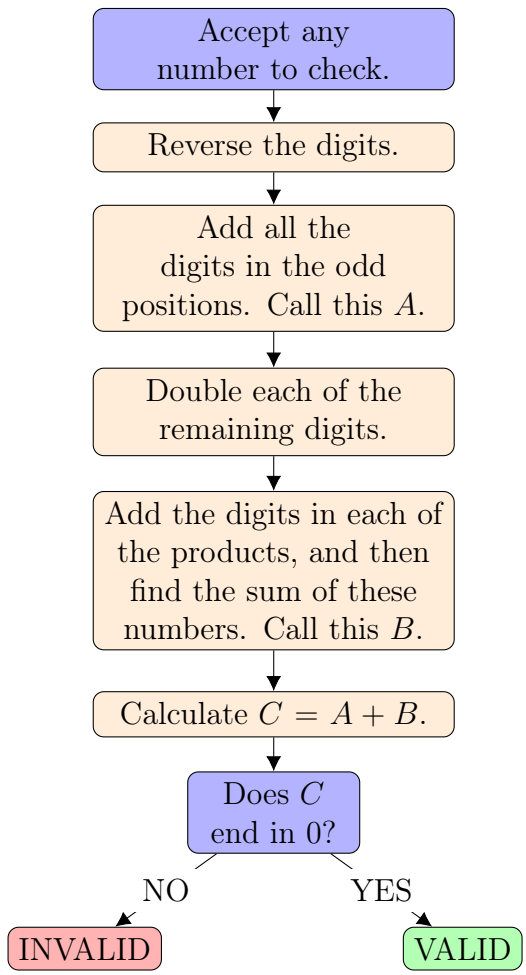
$A = 2 + 7 + 3$
$= 12$

$2 \times 9 = 18$
$2 \times 5 = 10$
$2 \times 1 = 2$

$B = (1 + 8) + (1 + 0) + 2$
$= 9 + 1 + 2$
$= 12$

$C = 12 + 12 = 24$

C does not end in zero.
The number is not valid.



Example 2:

Number: 1357987

Reversal: 7897531

$A = 7 + 9 + 5 + 1$
$= 22$

$2 \times 8 = 16$
$2 \times 7 = 14$
$2 \times 3 = 6$

$B = (1 + 6) + (1 + 4) + 6$
$= 7 + 5 + 6$
$= 18$

$C = 22 + 18 = 40$

C ends in zero.
The number is valid.

Suppose the number 4633 $RT0R$ 481 is a valid number when verified by the Luhn Algorithm, where R and T are each integers from 0 to 9 such that $R \leq T$.

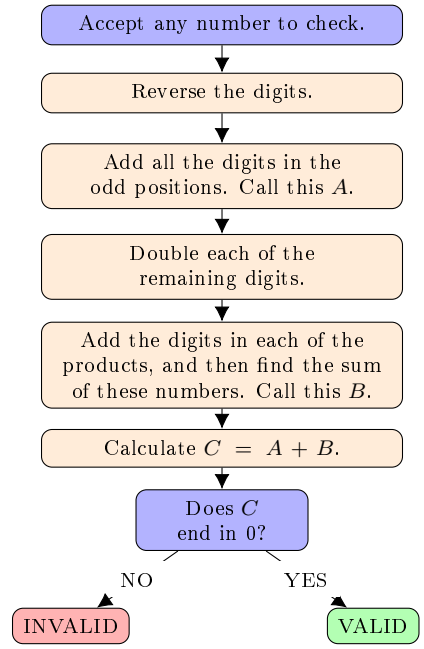
Determine all possible values of R and T.

Problem of the Week
Problem E and Solution
Is It Valid?

Problem

Debit and credit cards contain account numbers which consist of many digits. When purchasing items online, you are often asked to type in your account number. Because there are so many digits, it is easy to type the number incorrectly. The last digit of the number is a specially generated check digit which can be used to quickly verify the validity of the number. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid.

The steps performed in the Luhn Algorithm are outlined in the flowchart to the right.



Suppose the number 4633 $RT0R$ 481 is a valid number when verified by the Luhn Algorithm, where R and T are each integers from 0 to 9 such that $R \leq T$.

Determine all possible values of R and T.

Solution

When the digits of the number are reversed the resulting number is 184 $R0TR$ 3364. The sum of the digits in the odd positions is $A = 1 + 4 + 0 + R + 3 + 4 = 12 + R$.

When the digits in the remaining positions are doubled, the following products are obtained:

$$2 \times 8 = 16;\ 2 \times R = 2R;\ 2 \times T = 2T;\ 2 \times 3 = 6;\ \text{and } 2 \times 6 = 12$$

Let x represent the sum of the digits of $2R$ and y represent the sum of the digits of $2T$. When the digit sums from each of the products are added, the sum is:

$$B = (1 + 6) + x + y + 6 + (1 + 2) = 7 + x + y + 6 + 3 = x + y + 16$$

C is the sum of A and B, so $C = 12 + R + x + y + 16 = 28 + R + x + y$.

When an integer from 0 to 9 is doubled and the digits of the product are added together, what are the possible sums which can be obtained?

Original Digit (R or T)	0	1	2	3	4	5	6	7	8	9
Twice the Original Digit ($2R$ or $2T$)	0	2	4	6	8	10	12	14	16	18
The Sum of the Digits of $2R$ or $2T$ (x or y)	0	2	4	6	8	1	3	5	7	9

Notice that the sum of the digits of twice the original digit can only be an integer from 0 to 9 inclusive. It follows that the only values for x or y are the integers from 0 to 9.

The first row and the last row of this chart will be reprinted at the top of the next page as the results are used in the solution. We will use it to determine x from R and to determine T from y.

Original Digit (R or T)	0	1	2	3	4	5	6	7	8	9
The Sum of the Digits of $2R$ or $2T$ (x or y)	0	2	4	6	8	1	3	5	7	9

Since the number is valid, $C = 28 + R + x + y$ must end in zero.

What are the possible values to consider for C? Since the maximum value for each of R, x, and y is 9, the maximum value for $R + x + y$ is 27 and the maximum value for $C = 28 + R + x + y$ is 55. It follows that the only valid possibilities for C that end in zero are 30, 40, and 50. We will consider each of the three possibilities.

1. $C = 30$ and $R + x + y = 2$

   If $R = 0$, then $x = 0$, $y = 2$ and $T = 1$.
   There are no other valid possibilities for R so that $R + x + y = 2$.

   Therefore, this case produces one valid possibility for (R, T): $(0, 1)$.

2. $C = 40$ and $R + x + y = 12$

R	x	$y = 12 - R - x$	T	Valid or Invalid	(R, T)
0	0	12		not valid, $y > 9$	
1	2	9	9	valid, $R \leq T$	$(1, 9)$
2	4	6	3	valid, $R \leq T$	$(2, 3)$
3	6	3	6	valid, $R \leq T$	$(3, 6)$
4	8	0	0	not valid, $R > T$	
5	1	6	3	not valid, $R > T$	
6	3	3	6	valid, $R \leq T$	$(6, 6)$
7	5	0	0	not valid, $R > T$	
8	7	-3		not valid, $y < 0$	
9	9	-6		not valid, $y < 0$	

   Therefore, this case produces four valid possibilities for (R, T): $(1, 9)$, $(2, 3)$, $(3, 6)$ and $(6, 6)$.

3. $C = 50$ and $R + x + y = 22$

R	x	$y = 22 - R - x$	T	Valid or Invalid	(R, T)
0	0	22		not valid, $y > 9$	
1	2	19		not valid, $y > 9$	
2	4	16		not valid, $y > 9$	
3	6	13		not valid, $y > 9$	
4	8	10		not valid, $y > 9$	
5	1	16		not valid, $y > 9$	
6	3	13		not valid, $y > 9$	
7	5	10		not valid, $y > 9$	
8	7	7	8	valid, $R \leq T$	$(8, 8)$
9	9	4	2	not valid, $R > T$	

   Therefore, this case produces one valid possibility for (R, T): $(8, 8)$.

We have examined all possible values for C.

Therefore, there are six valid possibilities for (R, T): $(0, 1)$, $(1, 9)$, $(2, 3)$, $(3, 6)$, $(6, 6)$ and $(8, 8)$.

These correspond to the following six card numbers, which are indeed valid by the Luhn Algorithm: 4633 0100 481, 4633 1901 481, 4633 2302 481, 4633 3603 481, 4633 6606 481, and 4633 8808 481.

Problem of the Week

Problem E

Red Dog

We can take any word and rearrange all the letters to get another "word". These new "words" may be nonsensical. For example, you can rearrange the letters in $MATH$ to get $MTHA$.

Nalan wants to rearrange all the letters in $REDDOG$. However, she uses the following rules:

- the letters R, E, and D cannot be adjacent to each other and in that order, and
- the letters D, O, and G cannot be adjacent to each other and in that order.

For example, the "words" $DOGRED$, $DDOGRE$, $GDREDO$, and $DREDOG$ are examples of unacceptable words in this problem, but $DROEGD$ is acceptable.

How many different arrangements of the letters in $REDDOG$ can Nalan make if she follows these rules?

Problem of the Week
Problem E and Solution
Red Dog

Problem

We can take any word and rearrange all the letters to get another "word". These new "words" may be nonsensical. For example, you can rearrange the letters in $MATH$ to get $MTHA$.

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How many different arrangements of the letters in $REDDOG$ can Nalan make if she follows these rules?

Solution

We will find the total number of possible "words" Nalan can make, and then exclude those "words" which don't follow the rules (i.e. those which contain RED or DOG (or both)).

1. Determine the total number of "words" formed using 2 Ds, 1 E, 1 G, 1 O, and 1 R.

   First, place the E in 6 possible positions. Then, for each of the 6 possible placements of the E, there are 5 ways to place the G. There are then $6 \times 5 = 30$ ways to place the E and the G. Then, for each of the 30 possible placements of the E and G, there are 4 ways to place the O. There are then $30 \times 4 = 120$ ways to place the E, the G, and the O. Then, for each of the 120 possible placements of the E, G, and O, there are 3 ways to place the R. There are then $120 \times 3 = 360$ ways to place the E, the G, the O, and the R.

   For each of the 360 ways to place the E, G, O, and R, the 2 Ds must go in the remaining two empty spaces in 1 way. Therefore, there are $360 \times 1 = 360$ ways to place the E, the G, the O, the R, and the 2 Ds.

   Thus, there are 360 possible "words" that Nalan can make.

2. Determine how many "words" contain RED.

   There are 4 ways to place the word RED in the six spaces. The word RED could start in the first, second, third, or fourth position.

   $$R\ E\ D\ \_\ \_\ \_ \qquad \_\ R\ E\ D\ \_\ \_ \qquad \_\ \_\ R\ E\ D\ \_ \qquad \_\ \_\ \_\ R\ E\ D$$

   For each placement of the word RED, there are 6 ways to place the letters of the word DOG in the remaining three spaces: DOG, DGO, GDO, GOD, ODG and OGD. So there are $4 \times 6 = 24$ "words" containing RED.

3. Determine how many "words" contain DOG.

   There are 4 ways to place the word DOG in the six spaces. The word DOG could start in the first, second, third, or fourth position.

   $$D\ O\ G\ \_\ \_\ \_ \qquad \_\ D\ O\ G\ \_\ \_ \qquad \_\ \_\ D\ O\ G\ \_ \qquad \_\ \_\ \_\ D\ O\ G$$

   For each placement of the word DOG, there are 6 ways to place the letters of the word RED in the remaining three spaces: DER, DRE, EDR, ERD, RDE and RED. So there are $4 \times 6 = 24$ "words" containing DOG.

4. Determine how many "words" contain both RED and DOG.

   There are 4 "words" that contain both RED and DOG. They are as follows.

   $$REDDOG \qquad DOGRED \qquad REDOGD \qquad DREDOG$$

   These 4 "words" have been double-counted, as they would have been counted in both step 2 and step 3.

Thus in total, there are $24 + 24 - 4 = 44$ "words" that contain RED or DOG (or both). Since there are 360 possible "words" Nalan can make, we can subtract 44 from this to determine the number of these "words" that do <u>not</u> contain RED or DOG (or both).

Therefore, $360 - 44 = 316$ "words" can be formed in which the letters R, E and D are not adjacent to each other and in that order and the letters D, O and G are not adjacent to each other and in that order.

Problem of the Week

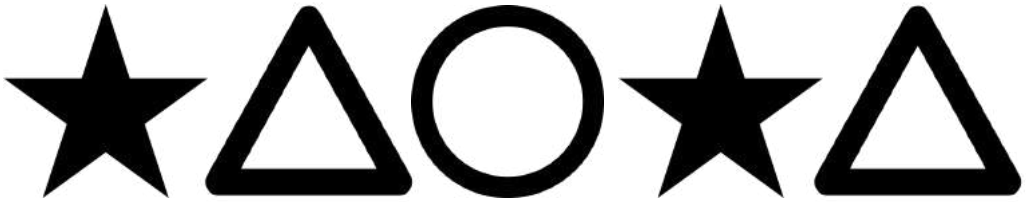
Problem E

A Small Subset

There are 90 000 five-digit positive integers. However, only some of these five-digit integers satisfy the following conditions:

- the middle digit is 0,
- the ten thousands digit and the tens digit are equal,
- the thousands digit and the ones (units) digit are equal, and
- the number has exactly 5 prime factors. All of these prime factors are odd and none are repeated.

Determine all five-digit positive integers that satisfy the given conditions.



Problem of the Week
Problem E and Solution
A Small Subset

Problem

There are 90 000 five-digit positive integers. However, only some of these five-digit integers satisfy the following conditions:

- the middle digit is 0,
- the ten thousands digit and the tens digit are equal,
- the thousands digit and the ones (units) digit are equal, and
- the number has exactly 5 prime factors. All of these prime factors are odd and none are repeated.

Determine all five-digit positive integers that satisfy the given conditions.

Solution

Solution 1

Let $st0st$ represent a five-digit positive integer satisfying the conditions. Notice that

$$st0st = st(1000) + st = st(1000 + 1) = st(1001)$$

This means that the number $st0st$ is divisible by 1001, which is the product of the three odd prime factors 7, 11, and 13. So st is a two-digit number which is the product of two different odd prime factors none of which can be 7, 11 or 13. We will now generate all possible two-digit products, using odd prime factors other than 7, 11 and 13.

Prime Factor a	Prime Factor b	st $= a \times b$	Five Different Odd Primes	Product $st0st$
3	5	15	$3, 5, 7, 11, 13$	15015
3	17	51	$3, 7, 11, 13, 17$	51051
3	19	57	$3, 7, 11, 13, 19$	57057
3	23	69	$3, 7, 11, 13, 23$	69069
3	29	87	$3, 7, 11, 13, 29$	87087
3	31	93	$3, 7, 11, 13, 31$	93093
5	17	85	$5, 7, 11, 13, 17$	85085
5	19	95	$5, 7, 11, 13, 19$	95095

No other two-digit product of two different odd prime factors other than 7, 11 and 13 exists.

Therefore, there are 8 five-digit positive integers that satisfy the given conditions. The numbers are 15015, 51051, 57057, 69069, 87087, 93093, 85085 and 95095.

Solution 2

Let $st0st$ represent a five-digit positive integer satisfying the conditions.

A number is divisible by 11 if the difference between the sum of the digits in the even positions and the sum of the digits in the odd positions is a multiple of 11. (This problem can still be done without knowing this divisibility fact but the task is made simpler with it.) The sum of the digits in the even positions is $st0st$ is $t + s$. The sum of the digits in the odd positions is $s + 0 + t$ which simplifies to $s + t$. The difference of the two sums is $(t + s) - (s + t) = 0$ which is a multiple of 11. Therefore, $st0st$ is divisible by 11.

Only odd factors are used, so the product will be odd. This means that the product looks like $s10s1$, $s30s3$, $s50s5$, $s70s7$, or $s90s9$ where s is a digit from 1 to 9. So we begin to systematically look at the possibilities.

First, we will examine numbers that have 3 (and 11) as a factor. To be divisible by 3, the sum of the digits will be divisible by 3. To be divisible by 9, the sum of the digits will be divisible by 9. But if the number is divisible by 9, it is divisible by 3^2 and would have a repeated prime factor which is not allowed. So we want numbers divisible by 3 but not 9. The possibilities are as follows: 21021, 51051, 33033, 93093, 15015, 75075, 57057, 87087, 39039, and 69069. The sum of the digits of each of these numbers is divisible by 3 so each of the numbers are divisible by 3. The numbers 81081, 63063, 45045, 27027 and 99099 are divisible by 9 and have therefore been eliminated.

Now we examine the prime factorization of each of these numbers to see which numbers satisfy the conditions.

$$21021 = 3 \times 11 \times 637 = 3 \times 11 \times 7 \times 91 = 3 \times 11 \times 7 \times 7 \times 13$$

Since the prime factor 7 is repeated, this is not a valid number.

$$51051 = 3 \times 11 \times 1547 = 3 \times 11 \times 7 \times 221 = 3 \times 11 \times 7 \times 13 \times 17$$

Since there are 5 different odd prime factors, 51051 is a **valid number**.

$$33033 = 3 \times 11 \times 1001 = 3 \times 11 \times 7 \times 143 = 3 \times 11 \times 7 \times 11 \times 13$$

Since the prime factor 11 is repeated, this is not a valid number.

$$93093 = 3 \times 11 \times 2821 = 3 \times 11 \times 7 \times 403 = 3 \times 11 \times 7 \times 13 \times 31$$

Since there are 5 different odd prime factors, 93093 is a **valid number**.

$$15015 = 3 \times 11 \times 455 = 3 \times 11 \times 5 \times 91 = 3 \times 11 \times 5 \times 7 \times 13$$

Since there are 5 different odd prime factors, 15015 is a **valid number**.

$$75075 = 3 \times 11 \times 2275 = 3 \times 11 \times 5 \times 455 = 3 \times 11 \times 5 \times 5 \times 91 = 3 \times 11 \times 5 \times 5 \times 7 \times 13$$

Since the prime factor 5 is repeated and there are six prime factors, this is not a valid number.

$$57057 = 3 \times 11 \times 1729 = 3 \times 11 \times 7 \times 247 = 3 \times 11 \times 7 \times 13 \times 19$$

Since there are 5 different odd prime factors, 57057 is a **valid number**.

$$87087 = 3 \times 11 \times 2639 = 3 \times 11 \times 7 \times 377 = 3 \times 11 \times 7 \times 13 \times 29$$

Since there are 5 different odd prime factors, 87087 is a **valid number**.

$$39039 = 3 \times 11 \times 1183 = 3 \times 11 \times 7 \times 169 = 3 \times 11 \times 7 \times 13 \times 13$$

Since the prime factor 13 is repeated, this is not a valid number.

$$69069 = 3 \times 11 \times 2093 = 3 \times 11 \times 7 \times 299 = 3 \times 11 \times 7 \times 13 \times 23$$

Since there are 5 different odd prime factors, 69069 is a **valid number**.

Second, we will examine numbers that are divisible by 5 but not 3, since divisibility by 3 has been examined. If a number is divisible by 5, then it ends in 5 or 0. Since the number is odd, we can exclude any number ending in 0, leaving 25025, 35035, 55055, 65065, 85085 and 95095 as possible numbers. (15015, 45045, 75075 were examined above and have been excluded.)

Now we examine the prime factorization of each of these numbers to see which numbers satisfy the conditions.

$$25025 = 5 \times 11 \times 455 = 5 \times 11 \times 5 \times 91 = 5 \times 11 \times 5 \times 7 \times 13$$

Since the prime factor 5 is repeated, this is not a valid number.

$$35035 = 5 \times 11 \times 637 = 5 \times 11 \times 7 \times 91 = 5 \times 11 \times 7 \times 7 \times 13$$

Since the prime factor 7 is repeated, this is not a valid number.

$$55055 = 5 \times 11 \times 1001 = 5 \times 11 \times 7 \times 143 = 5 \times 11 \times 7 \times 11 \times 13$$

Since the prime factor 11 is repeated, this is not a valid number.

$$65065 = 5 \times 11 \times 1183 = 5 \times 11 \times 7 \times 169 = 5 \times 11 \times 7 \times 13 \times 13$$

Since the prime factor 13 is repeated, this is not a valid number.

$$85085 = 5 \times 11 \times 1547 = 5 \times 11 \times 7 \times 221 = 5 \times 11 \times 7 \times 13 \times 17$$

Since there are 5 different odd prime factors, 85085 is a **valid number**.

$$95095 = 5 \times 11 \times 1729 = 5 \times 11 \times 7 \times 247 = 5 \times 11 \times 7 \times 13 \times 19$$

Since there are 5 different odd prime factors, 95095 is a **valid number**.

Thirdly, we will look at numbers that are divisible by 7 but not divisible by 3 or 5. If we multiply 7 by the next four odd prime numbers we get $7 \times 11 \times 13 \times 17 \times 19 = 323\,323$, a six-digit number, so we are beyond all possible solutions.

Therefore, there are 8 positive five-digit integers that satisfy the given conditions. The numbers are 51051, 93093, 15015, 57057, 87087, 69069, 85085 and 95095.

Problem of the Week

Problem E

Can Drive On!

Bryn observed a real need in her community for canned food at the local food bank. On April 15, she decided to start a canned food drive, and hoped to collect 4000 cans of food by the end of June. She posted flyers and spread the word. She made the following observations.

Day Number	Total Number of Cans Collected Since Beginning of Drive	Increase from Previous Day
1	2	
2	5	3
3	12	7
4	23	11
5	38	15

Bryn noticed that the increases from one day to the next form an arithmetic sequence with first term 3 and common difference 4. (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ is an arithmetic sequence with four terms and common difference 2.)

Assuming that the pattern of daily increases continues, how many days would it take to collect at least 4000 cans of food?

The information on the next page *may* be helpful in solving the problem.

The sequence $3, 5, 7, 9$ is an arithmetic sequence with four terms and common difference 2. The term in position n is denoted t_n. For example, we say that $t_1 = 3$. The subscript 1 is the position of the term in the sequence and 3 is the value of the term.

The general term of an arithmetic sequence is $t_n = a + (n-1)d$, where a is the first term, d is the common difference, and n is the term number.

The sum, S_n, of the first n terms of an arithmetic sequence can be found using either $S_n = \frac{n}{2}(2a + (n-1)d)$ or $S_n = n\left(\frac{t_1 + t_n}{2}\right)$, where t_1 is the first term of the sequence and t_n is the nth term of the sequence.

For example, for the arithmetic sequence $3, 5, 7, 9$, we have $a = t_1 = 3$, $d = 2$, $t_4 = 9$, and $S_4 = 3 + 5 + 7 + 9 = 24$.

Also,

$$\begin{aligned}
\frac{4}{2}(2a + (4-1)d) &= \frac{4}{2}(2(3) + (4-1)2) \\
&= 2(12) \\
&= 24 \\
&= S_4
\end{aligned}$$

And,

$$\begin{aligned}
4\left(\frac{t_1 + t_4}{2}\right) &= 4\left(\frac{3+9}{2}\right) \\
&= 4(6) \\
&= 24 \\
&= S_4
\end{aligned}$$

Problem of the Week
Problem E and Solution
Can Drive On!

Problem

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Bryn noticed that the increases from one day to the next form an arithmetic sequence with first term 3 and common difference 4. (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ is an arithmetic sequence with four terms and common difference 2.) Assuming that the pattern of daily increases continues, how many days would it take to collect at least 4000 cans of food?

Solution

Solution 1

On day i, let the total number of cans collected since the beginning of the drive be a_i. That is, $a_1 = 2$, $a_2 = 5$, $a_3 = 12$, $a_4 = 23$, and $a_5 = 38$. We want to determine the day number, n, so that $a_n \geq 4000$.

Let the arithmetic sequence of the increases from day to day be $t_1,\ t_2,\ t_3,\ \cdots,\ t_n,\ \cdots$.

Then $t_1 = a_2 - a_1 = 5 - 2 = 3$, $t_2 = a_3 - a_2 = 12 - 5 = 7$, $t_3 = a_4 - a_3 = 23 - 12 = 11$, and $t_4 = a_5 - a_4 = 38 - 23 = 15$. This sequence of increases is arithmetic, with common difference $d = t_2 - t_1 = 7 - 3 = 4$. To determine t_5, we add the common difference 4 to t_4. So $t_5 = t_4 + 4 = 15 + 4 = 19$. Then $a_6 = a_5 + t_5 = 38 + 19 = 57$. This means that the total number of cans on day 6 is 57. We could continue generating these increases and number of cans on the next day until the goal is reached.

Let's take a closer look at the number of cans.

$a_1 = 2$
$a_2 = a_1 + t_1 = 2 + 3 = 5$
$a_3 = a_2 + t_2 = 2 + 3 + 7 = 12$
$a_4 = a_3 + t_3 = 2 + 3 + 7 + 11 = 23$
$a_5 = a_4 + t_4 = 2 + 3 + 7 + 11 + 15 = 38$
$a_6 = a_5 + t_5 = 2 + 3 + 7 + 11 + 15 + 19 = 57$

We can present the information in a table.

Day Number n	Total Number of Cans a_n	Daily Increase t_n	Difference of the Daily Increase d
1	2	3	
2	5	7	4
3	12	11	4
4	23	15	4
5	38	19	4
6	57		

Since the second difference is constant, we can represent the general term of the sequence of total cans with a quadratic function in n. Let $a_n = pn^2 + qn + r$, where p, q, r are constants.

For $n = 1$, $a_1 = 2 = p(1)^2 + q(1) + r$. Therefore, $p + q + r = 2$. (1)
For $n = 2$, $a_2 = 5 = p(2)^2 + q(2) + r$. Therefore, $4p + 2q + r = 5$. (2)
For $n = 3$, $a_3 = 12 = p(3)^2 + q(3) + r$. Therefore, $9p + 3q + r = 12$. (3)

Subtracting (1) from (2), $3p + q = 3$. (4)
Subtracting (2) from (3), $5p + q = 7$. (5)

Subtracting (4) from (5), $2p = 4$ and $p = 2$ follows.
Substituting $p = 2$ into (4), $3(2) + q = 3$ and $q = -3$ follows.

Substituting $p = 2,\ q = -3$ into (1), $2 - 3 + r = 2$ and $r = 3$ follows.

Therefore, $a_n = 2n^2 - 3n + 3$ is the general term of the sequence of total cans, in terms of n.

We want to find the value of n so that $2n^2 - 3n + 3 \geq 4000$. We will do so by first solving $2n^2 - 3n + 3 = 4000$ for n. Rearranging the equation, we obtain $2n^2 - 3n - 3997 = 0$. Using the quadratic formula, $n \approx -43.96$ or $n \approx 45.46$. Since n is the day number, $n > 0$. Therefore, $n \approx 45.46$, and it follows that on day 46 there would be over 4000 cans collected in total.

If we check when $n = 45$, $a_{45} = 3918$ cans, which is under the goal.

When $n = 46$, $a_{46} = 4097$ cans and the goal is achieved. Therefore, it would take 46 days to collect at least 4000 cans of food. There are over 46 days from April 15 to the end of June, so it is possible to achieve the goal if the pattern continues.

Two more solutions follow.

Solution 2

As in Solution 1, on day i, let the total number of cans collected since the beginning of the drive be a_i and let the sequence of daily increases be $t_1,\ t_2,\ t_3,\ \cdots,\ t_n,\ \cdots$. Then $t_1 = a_2 - a_1 = 3$ and $t_2 = a_3 - a_2 = 7$. Since the sequence of daily increases is arithmetic, the constant difference is $d = 7 - 3 = 4$. We can generate more terms: $t_3 = 11,\ t_4 = 15,\ t_5 = 19,\ \cdots$.

Each term in the sequence of daily increases is the difference between consecutive terms of the original sequence, so we get the following equations.

$$\begin{aligned} a_2 - a_1 &= t_1 \\ a_3 - a_2 &= t_2 \\ a_4 - a_3 &= t_3 \\ &\vdots \\ a_{n-2} - a_{n-3} &= t_{n-3} \\ a_{n-1} - a_{n-2} &= t_{n-2} \\ a_n - a_{n-1} &= t_{n-1} \end{aligned}$$

Adding these equations, we get

$$a_n - a_1 = t_1 + t_2 + t_3 + \cdots + t_{n-2} + t_{n-1} = S_{n-1} \quad (1)$$

as illustrated below.

$$\begin{array}{rrcrcl} & \cancel{a_2} & - & a_1 & = & t_1 \\ & \cancel{a_3} & - & \cancel{a_2} & = & t_2 \\ & \cancel{a_4} & - & \cancel{a_3} & = & t_3 \\ & \vdots & & \vdots & & \vdots \\ & \cancel{a_{n-2}} & - & \cancel{a_{n-3}} & = & t_{n-3} \\ & \cancel{a_{n-1}} & - & \cancel{a_{n-2}} & = & t_{n-2} \\ + & a_n & - & \cancel{a_{n-1}} & = & t_{n-1} \\ \hline & a_n & - & a_1 & = & t_1 + t_2 + t_3 + \cdots + t_{n-2} + t_{n-1} \quad (1) \end{array}$$

To find the sum $S_{n-1} = t_1 + t_2 + t_3 + \cdots + t_{n-2} + t_{n-1}$, we can use the formula $S_n = \frac{n}{2}[2a + (n-1)d]$ with $a = 3$ and $d = 4$ for $(n-1)$ terms. So,

$$\begin{aligned} S_{n-1} &= t_1 + t_2 + t_3 + \cdots + t_{n-2} + t_{n-1} \\ &= \frac{n-1}{2}[2a + ((n-1)-1)d] \\ &= \frac{n-1}{2}[2(3) + ((n-1)-1)(4)] \\ &= \frac{n-1}{2}[6 + (n-2)(4)] \\ &= \frac{n-1}{2}[6 + 4n - 8] \\ &= \frac{n-1}{2}[4n - 2] \\ &= (n-1)(2n-1) \end{aligned}$$

From (1), $a_n - a_1 = S_{n-1}$, so $a_n = S_{n-1} + a_1 = (n-1)(2n-1) + 2 = 2n^2 - 3n + 3$.

Therefore, $a_n = 2n^2 - 3n + 3$ is the general term of the sequence of total cans, in terms of n.

Then, as in Solution 1, we would find the value of n so that $2n^2 - 3n + 3 \geq 4000$. Without repeating the work here, we find that it would take 46 days to collect at least 4000 cans of food.

Solution 3

As in Solution 1, on day i, let the total number of cans collected since the beginning of the drive be a_i and let the sequence of daily increases be $t_1,\ t_2,\ t_3,\ \cdots,\ t_n,\ \cdots$. Then $t_1 = a_2 - a_1 = 3$ and $t_2 = a_3 - a_2 = 7$. Since the sequence of daily increases is arithmetic, the constant difference is $d = 7 - 3 = 4$. Using the formula for the general term of an arithmetic sequence, $t_n = a + (n-1)d$, the general term of the sequence of increases is $t_n = 3 + (n-1)(4) = 3 + 4n - 4 = 4n - 1$.

To generate the sequence of total cans, we start with the first term and add more terms from the arithmetic sequence of daily increases. For example,
$a_1 = 2$
$a_2 = 2 + t_1 = 2 + [4(1) - 1] = 2 + 3 = 5$
$a_3 = 2 + t_1 + t_2 = 2 + [4(1) - 1] + [4(2) - 1] = 2 + 3 + 7 = 12$

So,

$$\begin{aligned}
a_n &= 2 + t_1 + t_2 + t_3 + \cdots + t_{n-1} \\
&= 2 + [4(1) - 1] + [4(2) - 1] + [4(3) - 1] + \cdots + [4(n-1) - 1] \\
&= 2 + [4(1) + 4(2) + 4(3) + \cdots + 4(n-1)] + (n-1)(-1) \\
&= 2 + 4[1 + 2 + 3 + \cdots + (n-1)] - n + 1 \\
&= 3 - n + 4\left[\frac{(n-1)(n)}{2}\right] \qquad (1) \\
&= 3 - n + 2(n^2 - n) \\
&= 3 - n + 2n^2 - 2n \\
&= 2n^2 - 3n + 3
\end{aligned}$$

An explanation is provided here to show how (1) is obtained.
Notice that $1 + 2 + 3 + \cdots + (n-1)$ is an arithmetic sequence with $(n-1)$ terms, first term $b_1 = 1$, and last term $b_{n-1} = (n-1)$. Using the formula for the sum of the terms of an arithmetic sequence, $S_n = n\left(\frac{b_1 + b_n}{2}\right)$, we obtain

$$S_{n-1} = (n-1)\left(\frac{1 + (n-1)}{2}\right) = (n-1)\left(\frac{n}{2}\right) = \frac{(n-1)(n)}{2}$$

Therefore, $a_n = 2n^2 - 3n + 3$ is the general term of the sequence of total cans, in terms of n.

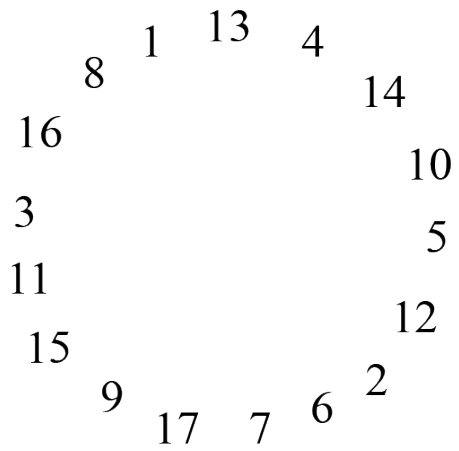
Then, as in Solution 1, we would find the value of n so that $2n^2 - 3n + 3 \geq 4000$. Without repeating the work here, we find that it would take 46 days to collect at least 4000 cans of food.

Problem of the Week

Problem E

Stand in a Circle

The numbers from 1 to 17 are arranged around a circle. One such arrangement is shown.



Explain why every possible arrangement of these numbers around a circle must have at least one group of three adjacent numbers whose sum is at least 27.

NOTE:
In solving the above problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

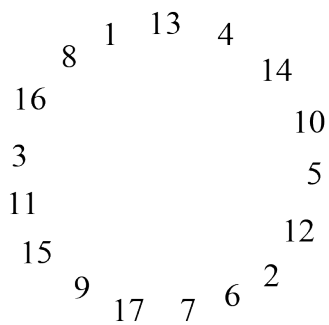
Problem of the Week

Problem E and Solution

Stand in a Circle

Problem

The numbers from 1 to 17 are arranged around a circle. One such arrangement is shown.



Explain why every possible arrangement of these numbers around a circle must have at least one group of three adjacent numbers whose sum is at least 27.

NOTE:
In solving the above problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

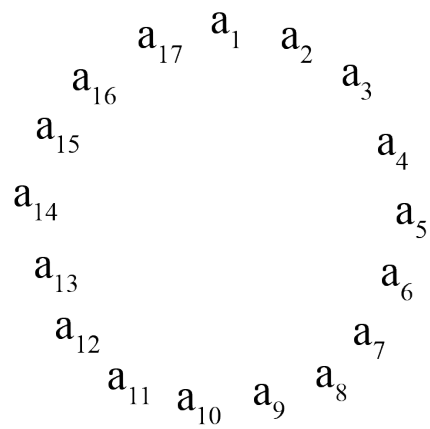
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Solution

We will use a *proof by contradiction* to explain why every possible arrangement of these numbers around a circle must have at least one group of three adjacent numbers whose sum is at least 27.

In general, to prove that a statement is true using a *proof by contradiction*, we first assume the statement is false. We then show this leads to a contradiction, which proves that our original assumption was wrong, and therefore the statement must be true.

First, we will assume that there exists an arrangement of the numbers from 1 to 17 around a circle where the sums of all groups of three adjacent numbers are less than 27. This arrangement is shown, where the variables $a_1,\ a_2,\ a_3, \ldots, a_{17}$ represent the numbers from 1 to 17, in some order, for this particular arrangement.



Now we will add up the sums of all groups of three adjacent numbers and call this value S.

$$\begin{aligned} S =& (a_1 + a_2 + a_3) + (a_2 + a_3 + a_4) + (a_3 + a_4 + a_5) \\ &+ (a_4 + a_5 + a_6) + (a_5 + a_6 + a_7) + (a_6 + a_7 + a_8) \\ &+ (a_7 + a_8 + a_9) + (a_8 + a_9 + a_{10}) + (a_9 + a_{10} + a_{11}) \\ &+ (a_{10} + a_{11} + a_{12}) + (a_{11} + a_{12} + a_{13}) + (a_{12} + a_{13} + a_{14}) \\ &+ (a_{13} + a_{14} + a_{15}) + (a_{14} + a_{15} + a_{16}) + (a_{15} + a_{16} + a_{17}) \\ &+ (a_{16} + a_{17} + a_1) + (a_{17} + a_1 + a_2) \end{aligned}$$

We can see that there are 17 groups of three adjacent numbers around the circle. Since each of these groups has a sum that is less than 27, we can conclude that S must be less than $17 \times 27 = 459$. So, $S < 459$.

Looking again at the value of S, we can see that each of $a_1,\ a_2,\ a_3, \ldots, a_{17}$ appears exactly three times. So,

$$\begin{aligned} S &= 3(a_1) + 3(a_2) + 3(a_3) + \cdots + 3(a_{17}) \\ &= 3(a_1 + a_2 + a_3 + \cdots + a_{17}) \end{aligned}$$

However, we know that $a_1 + a_2 + a_3 + \cdots + a_{17}$ is equal to the sum of all the numbers from 1 to 17, which is $\frac{17(18)}{2} = 153$. Therefore, $S = 3(153) = 459$.

But this is a contradiction, since we stated earlier that $S < 459$. It can't be possible that $S < 459$ and $S = 459$. Therefore, our original assumption that there exists an arrangement of the numbers from 1 to 17 around a circle where the sums of all groups of three adjacent numbers are less than 27 must be false. Thus, it follows that every possible arrangement of the numbers from 1 to 17 around a circle must have at least one group of three adjacent numbers whose sum is at least 27.

Geometry (G)

TAKE ME TO THE COVER

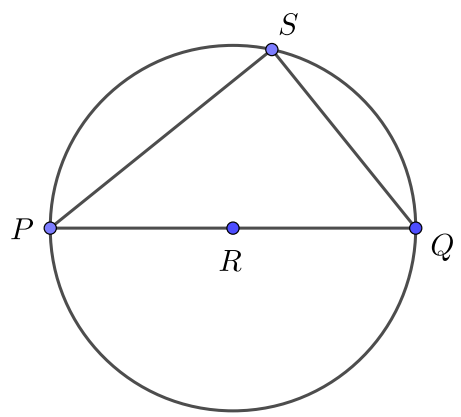
Problem of the Week

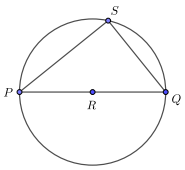
Problem E

Angled III

In the circle with centre R below, PQ is a diameter. Point S is a point on the circumference of the circle other than P or Q.

Determine the measure of $\angle PSQ$.





Problem of the Week

Problem E and Solution

Angled III

Problem

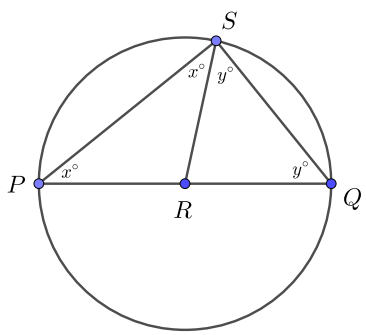
In the circle with centre R above, PQ is a diameter. Point S is a point on the circumference of the circle other than P or Q. Determine the measure of $\angle PSQ$.

Solution

Join S to the centre R. Since RP, RQ and RS are radii of the circle, $RP = RQ = RS$.

Since $RP = RS$, $\triangle PRS$ is isosceles and $\angle RPS = \angle RSP = x^\circ$.
Since $RQ = RS$, $\triangle QRS$ is isosceles and $\angle RQS = \angle RSQ = y^\circ$.

This new information is marked on the following diagram.



The angles in a triangle add to 180°, so in $\triangle PQS$

$$\begin{aligned}
\angle PSQ + \angle QPS + \angle PQS &= 180^\circ \\
(x^\circ + y^\circ) + x^\circ + y^\circ &= 180^\circ \\
2(x^\circ + y^\circ) &= 180^\circ \\
x^\circ + y^\circ &= 90^\circ
\end{aligned}$$

But $\angle PSQ = x^\circ + y^\circ$, so $\angle PSQ = 90^\circ$.

This result is often expressed as a theorem for circles:
An angle($\angle PSQ$) inscribed in a circle by a diameter (PQ) of the circle is 90°.

Problem of the Week

Problem E

The Area of the Year

In the diagram, $\triangle AB_1C_1$ is right-angled with $AB_1 = 2$ and $AC_1 = 5$. Lines AB_1 and AC_1 are extended and many more points are labelled at intervals of 1 unit, so that

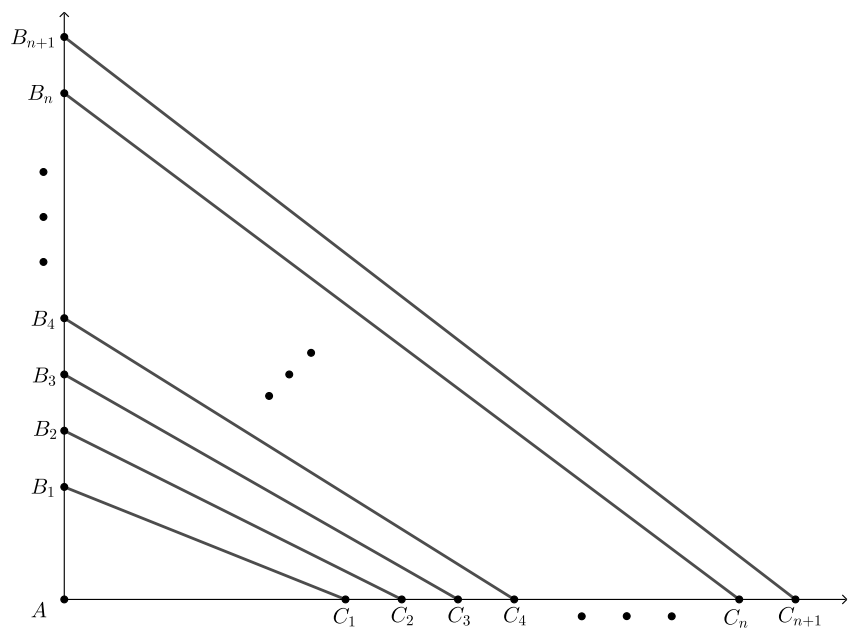
$$B_1B_2 = B_2B_3 = B_3B_4 = B_4B_5 = \cdots = 1, \text{ and}$$

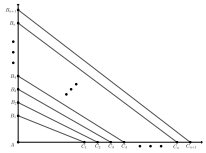
$$C_1C_2 = C_2C_3 = C_3C_4 = C_4C_5 = \cdots = 1.$$

In fact, $B_1B_j = j - 1$ and $C_1C_k = k - 1$ for any positive integers j and k.

For example, $B_1B_5 = 5 - 1 = 4$ and $C_1C_4 = 4 - 1 = 3$.

Determine the value of n so that the area of quadrilateral $B_nB_{n+1}C_{n+1}C_n$ is 2020. That is, determine the value of n so that the area of the quadrilateral with vertices B_n, B_{n+1}, C_{n+1}, and C_n is 2020.



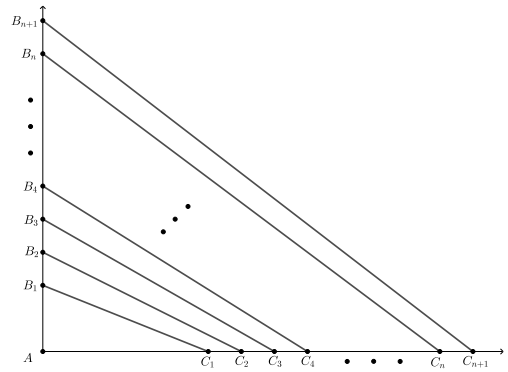


Problem of the Week

Problem E and Solution

The Area of the Year

Problem



In the diagram, $\triangle AB_1C_1$ is right-angled with $AB_1 = 2$ and $AC_1 = 5$. Lines AB_1 and AC_1 are extended and many more points are labelled at intervals of 1 unit, so that $B_1B_2 = B_2B_3 = B_3B_4 = B_4B_5 = \cdots = 1$, and $C_1C_2 = C_2C_3 = C_3C_4 = C_4C_5 = \cdots = 1$. In fact, $B_1B_j = j - 1$ and $C_1C_k = k - 1$ for any positive integers j and k. For example, $B_1B_5 = 5 - 1 = 4$ and $C_1C_4 = 4 - 1 = 3$. Determine the value of n so that the area of quadrilateral $B_nB_{n+1}C_{n+1}C_n$ is 2020.

Solution

Solution 1

In order to solve the problem, looking at the calculation of a specific area may prove helpful.

So let's determine the area of quadrilateral $B_4B_5C_5C_4$.

$$\begin{aligned}
\text{Area of quadrilateral } B_4B_5C_5C_4 &= \text{Area } \triangle B_5AC_5 - \text{Area } \triangle B_4AC_4 \\
&= \frac{1}{2}(AB_5)(AC_5) - \frac{1}{2}(AB_4)(AC_4) \\
&= \frac{1}{2}(2 + (5-1))(5 + (5-1)) - \frac{1}{2}(2 + (4-1))(5 + (4-1)) \\
&= \frac{1}{2}(6)(9) - \frac{1}{2}(5)(8) \\
&= 27 - 20 \\
&= 7 \text{ units}^2
\end{aligned}$$

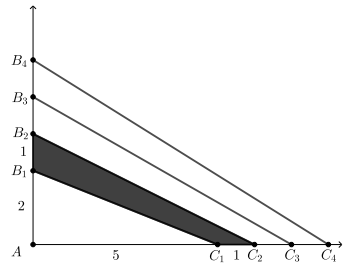
We will solve the problem by following what we did in the above example.

$$\begin{aligned}
\text{Area of quad. } B_nB_{n+1}C_{n+1}C_n &= \text{Area } \triangle B_{n+1}AC_{n+1} - \text{Area } \triangle B_nAC_n \\
2020 &= \frac{1}{2}(AB_{n+1})(AC_{n+1}) - \frac{1}{2}(AB_n)(AC_n) \\
2020 &= \frac{1}{2}(2 + ((n+1)-1))(5 + ((n+1)-1)) - \frac{1}{2}(2 + (n-1))(5 + (n-1)) \\
2020 &= \frac{1}{2}(2+n)(5+n) - \frac{1}{2}(1+n)(4+n) \\
4040 &= (2+n)(5+n) - (1+n)(4+n) \qquad \text{multiplying by 2} \\
4040 &= n^2 + 7n + 10 - (n^2 + 5n + 4) \\
4040 &= n^2 + 7n + 10 - n^2 - 5n - 4 \\
4040 &= 2n + 6 \\
4034 &= 2n \\
2017 &= n
\end{aligned}$$

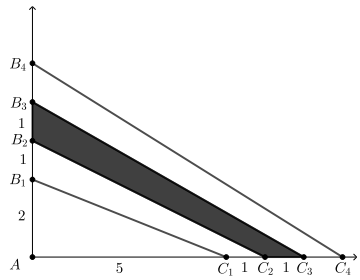
Therefore, the value of n is 2017.

Solution 2

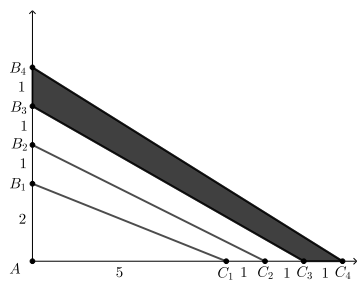
In this solution we look for a pattern in the area calculations.



$$\begin{aligned}
\text{Area of quad. } B_1B_2C_2C_1 &= \text{Area } \triangle B_2AC_2 - \text{Area } \triangle B_1AC_1 \\
&= \frac{1}{2}(AB_2)(AC_2) - \frac{1}{2}(AB_1)(AC_1) \\
&= \frac{1}{2}(3)(6) - \frac{1}{2}(2)(5) \\
&= 9 - 5 \\
\text{Area of first quad.} &= 4 \text{ units}^2
\end{aligned}$$



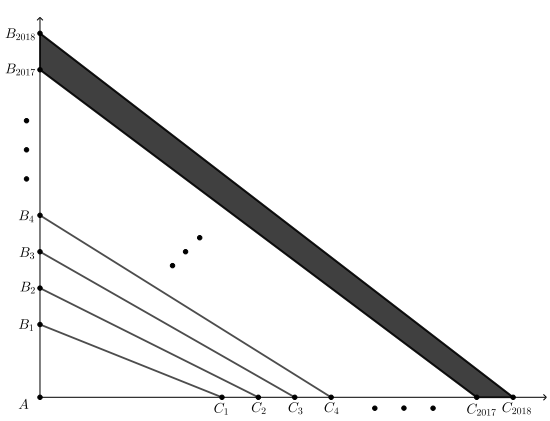
$$\begin{aligned}
\text{Area of quad. } B_2B_3C_3C_2 &= \text{Area } \triangle B_3AC_3 - \text{Area } \triangle B_2AC_2 \\
&= \frac{1}{2}(AB_3)(AC_3) - \frac{1}{2}(AB_2)(AC_2) \\
&= \frac{1}{2}(4)(7) - \frac{1}{2}(3)(6) \\
&= 14 - 9 \\
\text{Area of second quad.} &= 5 \text{ units}^2
\end{aligned}$$



$$\begin{aligned}
\text{Area of quad. } B_3B_4C_4C_3 &= \text{Area } \triangle B_4AC_4 - \text{Area } \triangle B_3AC_3 \\
&= \frac{1}{2}(AB_4)(AC_4) - \frac{1}{2}(AB_3)(AC_3) \\
&= \frac{1}{2}(5)(8) - \frac{1}{2}(4)(7) \\
&= 20 - 14 \\
\text{Area of third quad.} &= 6 \text{ units}^2
\end{aligned}$$

A possible pattern has emerged. The area of the quadrilateral appears to be three more than the position of the quadrilateral on the stack of consecutive quadrilaterals. If we want the area to be 2020, then it should be the 2017th quadrilateral. That is, it should be the quadrilateral with vertices $B_{2017}B_{2018}C_{2018}C_{2017}$. Therefore, the value of n is 2017.

We can verify this value of n using the area calculation. Recall from the problem statement, $B_1B_j = j - 1$ and $C_1C_k = k - 1$ for any positive integers j and k.
So, $B_1B_{2017} = 2017 - 1 = 2016$. Then $AB_{2017} = AB_1 + B_1B_{2017} = 2 + 2016 = 2018$.
Since $AB_{2018} = AB_{2017} + 1$, it follows that $AB_{2018} = 2019$.
Also, $C_1C_{2017} = 2017 - 1 = 2016$. Then $AC_{2017} = AC_1 + C_1C_{2017} = 5 + 2016 = 2021$.
Since $AC_{2018} = AC_{2017} + 1$, it follows that $AC_{2018} = 2022$.



$$\begin{aligned}
&\text{Area of quadrilateral } B_{2017}B_{2018}C_{2018}C_{2017} \\
&= \text{Area } \triangle B_{2018}AC_{2018} - \text{Area } \triangle B_{2017}AC_{2017} \\
&= \frac{1}{2}(AB_{2018})(AC_{2018}) - \frac{1}{2}(AB_{2017})(AC_{2017}) \\
&= \frac{1}{2}(2019)(2022) - \frac{1}{2}(2018)(2021) \\
&= 2\,041\,209 - 2\,039\,189 \\
&= 2020 \text{ units}^2
\end{aligned}$$

Problem of the Week

Problem E

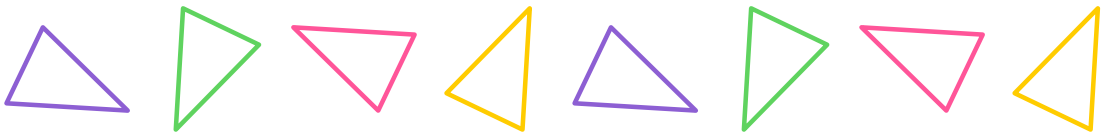
Terry's Triangles

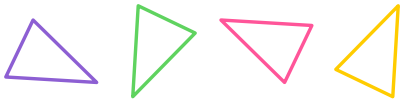
Terry is drawing isosceles triangles with side lengths a, b, and c such that

$$a = y - x$$
$$b = x + z$$
$$c = y - z$$

Where x, y, and z are positive integers and $x + y + z < 10$.

Find all the possible triples (a, b, c) that satisfy this.





Problem of the Week
Problem E and Solution
Terry's Triangles

Problem

Terry is drawing isosceles triangles with side lengths a, b, and c such that

$$a = y - x$$
$$b = x + z$$
$$c = y - z$$

Where x, y, and z are positive integers and $x + y + z < 10$.

Find all the possible triples (a, b, c) that satisfy this.

Solution

In an isosceles triangle, two sides must have equal length. So we need to consider three cases: $a = b$, $b = c$, and $a = c$. Also, in order for a, b, and c to represent side lengths of a triangle, they must be positive numbers and the sum of any two side lengths must be greater than the other side length.

Case 1: $a = b$

If $a = b$, then $y - x = x + z$, so $y = 2x + z$. We can make a table of all the values of x, y, and z that satisfy this equation as well as $x + y + z < 10$, and then find the corresponding values of a, b, and c and check if they are valid side lengths.

x	y	z	a	b	c	Valid?
1	3	1	2	2	2	Yes
1	4	2	3	3	2	Yes
1	5	3	4	4	2	Yes
2	5	1	3	3	4	Yes

Case 2: $b = c$

If $b = c$, then $x + z = y - z$, so $y = x + 2z$. As in Case 1, we can write the possible values of x, y, z, a, b, and c in a table.

x	y	z	a	b	c	Valid?
1	3	1	2	2	2	Yes
2	4	1	2	3	3	Yes
3	5	1	2	4	4	Yes
1	5	2	4	3	3	Yes

Case 3: $a = c$

If $a = c$, then $y - x = y - z$, so $x = z$. As in previous cases, we can write the possible values of x, y, z, a, b, and c in a table.

x	y	z	a	b	c	Valid?
1	1	1	0	2	0	No (a and c are not positive)
1	2	1	1	2	1	No ($a + c \not> b$)
1	3	1	2	2	2	Yes
1	4	1	3	2	3	Yes
1	5	1	4	2	4	Yes
1	6	1	5	2	5	Yes
1	7	1	6	2	6	Yes
2	1	2	-1	4	-1	No (a and c are not positive)
2	2	2	0	4	0	No (a and c are not positive)
2	3	2	1	4	1	No ($a + c \not> b$)
2	4	2	2	4	2	No ($a + c \not> b$)
2	5	2	3	4	3	Yes
3	1	3	-2	6	-2	No (a and c are not positive)
3	2	3	-1	6	-1	No (a and c are not positive)
3	3	3	0	6	0	No (a and c are not positive)
4	1	4	-3	8	-3	No (a and c are not positive)

Therefore, there are 12 possible triples (a, b, c). They are listed below.

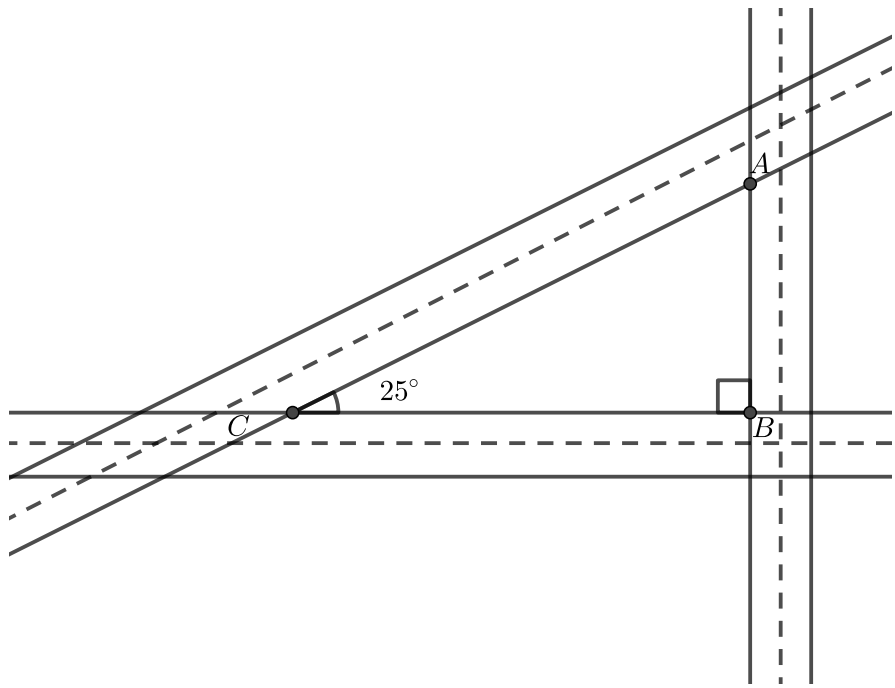
$$(2,2,2) \quad (3,3,2) \quad (4,4,2) \quad (3,3,4)$$
$$(2,3,3) \quad (2,4,4) \quad (4,3,3)$$
$$(3,2,3) \quad (4,2,4) \quad (5,2,5) \quad (6,2,6) \quad (3,4,3)$$

Problem of the Week

Problem E

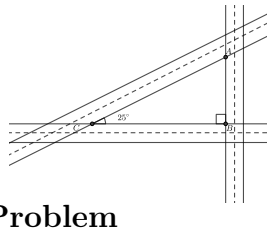
Roads All Around

Tima owns a triangular parcel of land that is created by three intersecting roads, as shown. Two of the roads meet at a right angle and two of the roads intersect at a 25° angle. If the perimeter of the triangular parcel of land is 1000 m, what is its area to the nearest 100 m^2?



Recall: For any acute angle, θ, of any right-angled triangle, we define the following:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}},\ \cos\theta = \frac{\text{adj}}{\text{hyp}},\ \tan\theta = \frac{\text{opp}}{\text{adj}}$$



Problem of the Week

Problem E and Solution

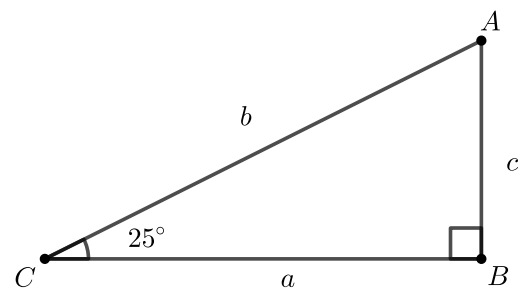
Roads All Around

Problem

Tima owns a triangular parcel of land that is created by three intersecting roads, as shown. Two of the roads meet at a right angle and two of the roads intersect at a 25° angle. If the perimeter of the triangular parcel of land is 1000 m, what is its area to the nearest 100 m^2?

Solution

We will label the diagram as shown to the right.



We know the following:

$a + b + c = 1000$ (1)

$\frac{c}{b} = \sin(25°)$, and so $c = b\sin(25°)$ (2)

$\frac{a}{b} = \cos(25°)$, and so $a = b\cos(25°)$ (3)

Substituting (2) and (3) into (1) we get:

$$\begin{aligned} b\cos(25°) + b + b\sin(25°) &= 1000 \\ b(\cos(25°) + 1 + \sin(25°)) &= 1000 \\ b &= \frac{1000}{\cos(25°) + 1 + \sin(25°)} \\ b &\approx 429.38 \text{ m} \end{aligned}$$

Now, since $a = b\sin(25°)$ and $c = b\cos(25°)$, the area of the triangle is

$$\begin{aligned} \frac{ac}{2} &= \frac{(b\sin(25°))(b\cos(25°))}{2} \\ &= \frac{b^2\sin(25°)\cos(25°)}{2} \\ &\approx \frac{(429.38)^2\sin(25°)\cos(25°)}{2} \\ &= 35\,308.4 \text{ m}^2 \end{aligned}$$

Therefore, to the nearest 100 m^2, the area of the triangle is 35 300 m^2.

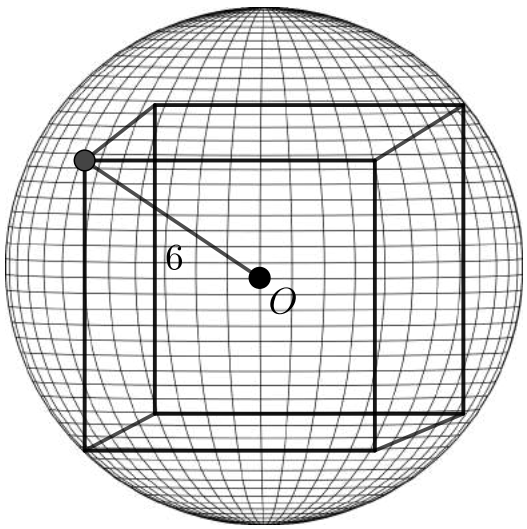
Problem of the Week

Problem E

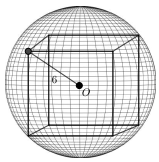
All Around the Cube

A cube is said to be *inscribed* in a sphere when all the vertices of the cube are on the surface of the sphere.

In the diagram below, the cube is inscribed in the sphere with centre O. If the radius of the sphere is 6 cm, determine the volume of the cube.



NOTE:
In solving the above problem, it may be helpful to use the fact that if a cube is inscribed in a sphere with centre O, then the cube will also have centre O.



Problem of the Week

Problem E and Solution

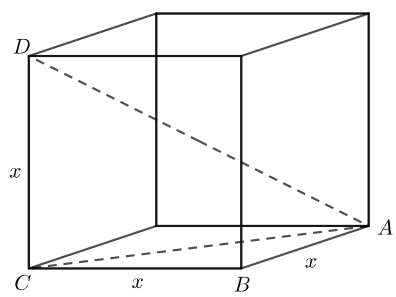
All Around the Cube

Problem

A cube is said to be *inscribed* in a sphere when all the vertices of the cube are on the surface of the sphere. In the diagram below, the cube is inscribed in the sphere with centre O. If the radius of the sphere is 6 cm, determine the volume of the cube.

Solution

Label four of the vertices of the cube A, B, C, D, as shown in the diagram. Let x represent the side length of the cube. Then $AB = BC = CD = x$.



The diagonals of a cube intersect at a point such that the distance from the intersection point to each vertex is equal. Since each vertex of the cube is on the sphere, the diagonal of the cube, AD, is equal in length to the diameter of the sphere. Therefore, $AD = 2(6) = 12$ cm.

Each face of a cube is a square, so $\angle ABC = 90^\circ$. Using the Pythagorean Theorem in $\triangle ABC$,

$$AC^2 = AB^2 + BC^2 = x^2 + x^2 = 2x^2$$

In a cube the sides are perpendicular to the base. In particular, DC is perpendicular to the base and it follows that $DC \perp AC$. Therefore $\triangle DCA$ is a right-angled triangle. Using the Pythagorean Theorem in $\triangle DCA$,

$$AD^2 = AC^2 + CD^2 = 2x^2 + x^2 = 3x^2$$

But $AD = 12$, so $AD^2 = 144$. Then,

$$\begin{aligned} 3x^2 &= 144 \\ x^2 &= 48 \\ x &= 4\sqrt{3}, \qquad \text{since } x > 0 \end{aligned}$$

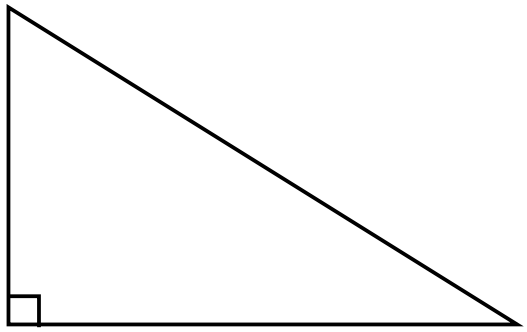
Therefore, the volume of the cube is $x^3 = (4\sqrt{3})^3 = 192\sqrt{3}$ cm^3.

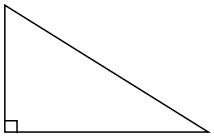
Problem of the Week

Problem E

That's Right

The side lengths of a right-angled triangle are all positive two-digit integers. If the digits representing the length of the hypotenuse are the digits of one of the side lengths written in the reverse order, find all the possible lengths of the hypotenuse.





Problem of the Week

Problem E and Solution

That's Right

Problem

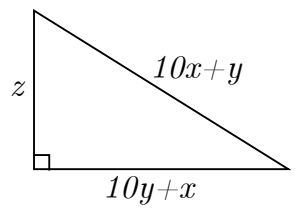
The side lengths of a right-angled triangle are all positive two-digit integers. If the digits representing the length of the hypotenuse are the digits of one of the side lengths written in the reverse order, find all the possible lengths of the hypotenuse.

Solution

Let x represent the tens digit of the hypotenuse such that x is an integer from 1 to 9. Let y represent the units digit of the hypotenuse such that y is an integer from 1 to 9. Then the length of the hypotenuse is $10x + y$.

Since one of the sides has the same digits as the hypotenuse in reverse order, the length of this side is $10y + x$.

Let the third side be z such that z is a two digit integer.



$$\begin{aligned}
\text{Using the Pythagorean Theorem:} \quad (10y+x)^2 + z^2 &= (10x+y)^2 \\
\text{Expanding:} \quad 100y^2 + 20xy + x^2 + z^2 &= 100x^2 + 20xy + y^2 \\
\text{Rearranging:} \quad z^2 &= 99x^2 - 99y^2 \\
\text{Factoring:} \quad z^2 &= 99(x-y)(x+y)
\end{aligned}$$

Since z^2 is a perfect square, $99(x+y)(x-y)$ must also be a perfect square.
But $99(x+y)(x-y) = 9(11)(x+y)(x-y)$. So, to be a perfect square, $(x+y)(x-y)$ must contain a factor of 11. Since x and y are each integers from 1 to 9, $x - y$ cannot be equal to 11 and $x + y$ cannot be greater than 18, and so we must have $x + y = 11$. Now, for $9(11)(x+y)(x-y) = 9(11)(11)(x-y)$ to be a perfect square, $x - y$ must be a perfect square.
Since x and y are integers from 1 to 9, there are three possibilities for $x - y$ that give a perfect square; $x - y = 1$, $x - y = 4$ or $x - y = 9$. These are the three possibilities:
$(x+y)(x-y) = 11(1)$ or $(x+y)(x-y) = 11(4)$ or $(x+y)(x-y) = 11(9)$.

Case 1: $(x+y)(x-y) = 11(1)$
If $x + y = 11$ and $x - y = 1$, we solve the system of equations obtaining $x = 6$ and $y = 5$. This gives a hypotenuse of $10x + y = 65$ and second side $10y + x = 56$. Then solving for z, $z^2 = 99(x+y)(x-y) = 99(11)(1) = 1089$ and $z = 33$. This solution is easily confirmed but is it the only solution?

Case 2: $(x+y)(x-y) = 11(4)$
If $x + y = 11$ and $x - y = 4$, we solve the system of equations obtaining $x = 7.5$ and $y = 3.5$. But x and y are not integers so this solution is inadmissible.

Case 3: $(x+y)(x-y) = 11(9)$
If $x + y = 11$ and $x - y = 9$, we solve the system of equations obtaining $x = 10$ and $y = 1$. But x must be an integer from 1 to 9 so this solution is inadmissible.

Therefore, the only solution is a hypotenuse of length 65.

Problem of the Week
Problem E
Cupid's Arrow

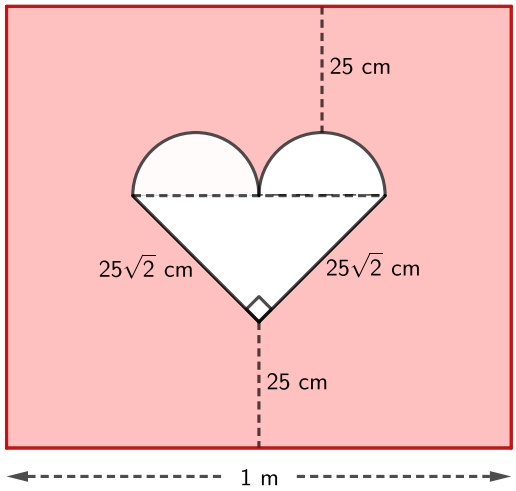
At a Valentine's dance, contestants can participate in a game. They are blindfolded and spun around. They then try to place an arrow on a white heart on a red gameboard. (See the diagram below.)

The heart was constructed by attaching two white semi-circles onto the hypotenuse of an isosceles right-angled triangle. Each semi-circle has the same diameter, equal to half the length of the hypotenuse. The heart was then pasted onto a large rectangular sheet of red paper such that the hypotenuse of the triangle is parallel to the base of the rectangle. We know the following measurements:

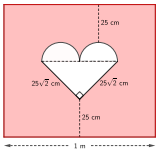
- The distance from the top of a semi-circle to the top of the rectangle is 25 cm.
- The distance from the bottom vertex of triangle to the bottom of the rectangle is 25 cm.
- The length of each equal side of the triangle is $25\sqrt{2}$ cm.
- The length of the base of the rectangle is 1 m.

(The dashed lines, the length measurements and the right angle symbol will not actually be on the finished gameboard.)

If a contestant places their arrow randomly somewhere on the gameboard, what is the probability that it will land on the white heart?



NOTE: To solve the problem, it may be helpful to use the following fact:
In an isosceles triangle, the line joining the vertex opposite the unequal side to the midpoint of the unequal side will be perpendicular to the unequal side.



Problem of the Week
Problem E and Solution
Cupid's Arrow

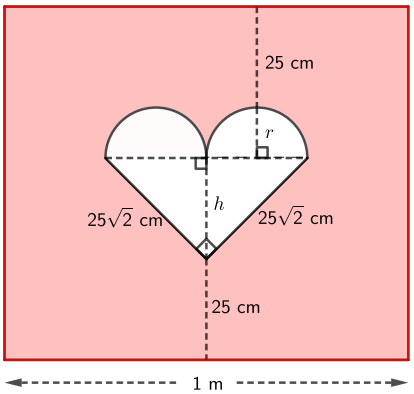
Problem

At a Valentine's dance, contestants can participate in a game. They are blindfolded and spun around. They then try to place an arrow on a white heart on a red gameboard. (See the diagram below.) The heart was constructed by attaching two white semi-circles onto the hypotenuse of an isosceles right-angled triangle. Each semi-circle has the same diameter, equal to half the length of the hypotenuse. The heart was then pasted onto a large rectangular sheet of red paper such that the hypotenuse of the triangle is parallel to the base of the rectangle. We know the following measurements:

- The distance from the top of a semi-circle to the top of the rectangle is 25 cm.
- The distance from the bottom vertex of triangle to the bottom of the rectangle is 25 cm.
- The length of each equal side of the triangle is $25\sqrt{2}$ cm.
- The length of the base of the rectangle is 1 m.

(The dashed lines, the length measurements and the right angle symbol will not actually be on the finished gameboard.) If a contestant places their arrow randomly somewhere on the gameboard, what is the probability that it will land on the white heart?

Solution



To find the probability of placing the arrow in the white heart, we need to find the area of the heart and the area of the rectangle. We will label a radius of a semi-circle with r and the distance from the vertex of the right-angled triangle to the midpoint of the hypotenuse with h. Using the helpful fact, we know that this line will be perpendicular to the hypotenuse. Therefore, the height of the rectangle is $25 + r + h + 25$.
We now need to find the values of r and h.

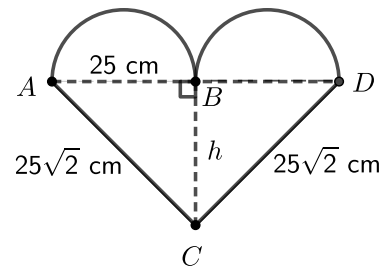
We will first find the value of r.

In the right-angled triangle, we know the the two equal sides are $25\sqrt{2}$ and the hypotenuse is equal to the sum of the two diameters of the semi-circles, which is $4r$.
Since the triangle is a right-angled triangle, we can use the Pythagorean Theorem to find the value of r.

$$\begin{aligned}
(4r)^2 &= (25\sqrt{2})^2 + (25\sqrt{2})^2 \\
16r^2 &= 1250 + 1250 \\
16r^2 &= 2500 \\
r^2 &= 156.25 \\
r &= 12.5
\end{aligned}$$

We can now find h.

Let's look at the heart. We will label the original triangle $\triangle ADC$ and let B be the midpoint of AD. From above, we know that $\angle ABC = 90^\circ$. We also know $AC = 25\sqrt{2}$, $AB = 2r = 25$, and $BC = h$.



Since $\triangle ABC$ is a right-angled triangle, we can use the Pythagorean Theorem to find the value of h.

$$\begin{aligned} h^2 &= (25\sqrt{2})^2 - 25^2 \\ h^2 &= 1250 - 625 \\ h^2 &= 625 \\ h &= 25, \quad \text{since } h > 0 \end{aligned}$$

We can now find the area of the rectangle and the area of the heart.

For the rectangle, the base is 1 m or 100 cm.
The height is $25 + r + h + 25 = 25 + 12.5 + 25 + 25 = 87.5$ cm.
Therefore, the area of the rectangle is $100(87.5) = 8750$ cm^2.

For the heart, the area is the total of the area of the two semi-circles plus the area of $\triangle ADC$. The two semi-circles make up a circle of radius 12.5 cm. For $\triangle ADC$, the base is $AD = 4r = 50$ and the height is $h = 25$.
Therefore, the area of the heart is $\pi(12.5)^2 + \dfrac{(50)(25)}{2} = (156.25\pi + 625)$ cm^2.

Therefore, the probability that the arrow will land on the white heart is

$$\frac{\text{area of heart}}{\text{area of rectangle}} = \frac{156.25\pi + 625}{8750} \approx 0.13$$

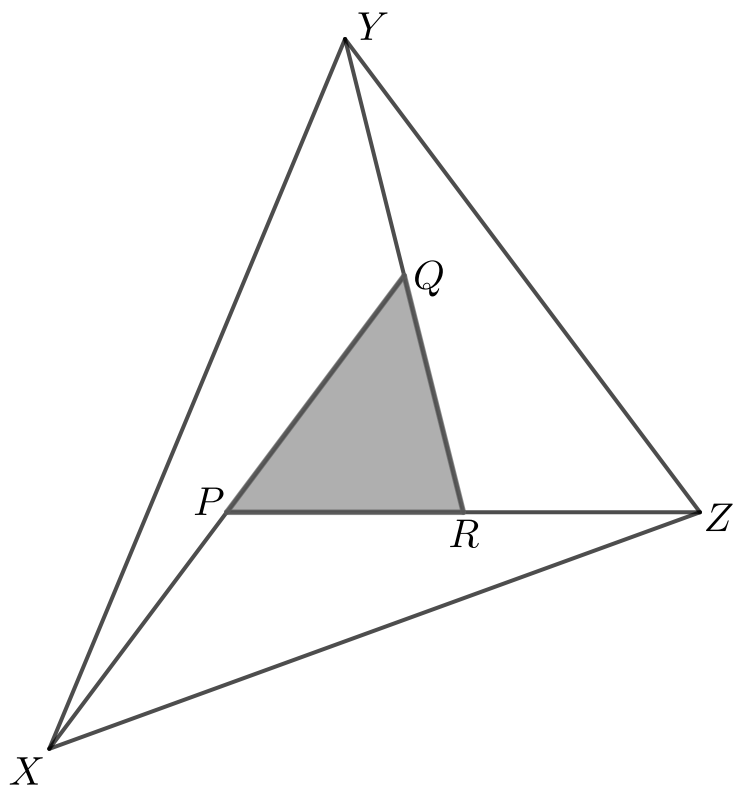
Problem of the Week

Problem E

Good Extensions

The sides of $\triangle PQR$ are extended to create $\triangle XYZ$ as follows: side RQ is extended to Y so that $RQ = QY$, side QP is extended to X so that $QP = PX$, and side PR is extended to Z so that $PR = RZ$.

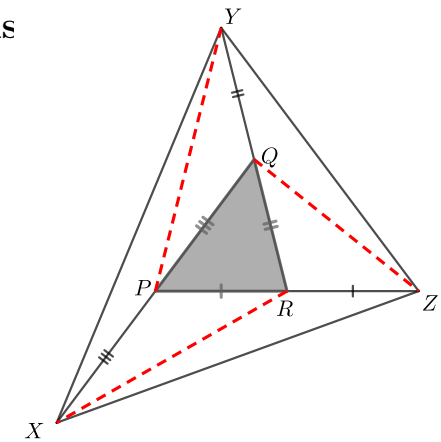
If the area of $\triangle XYZ = 1197$, determine the area of $\triangle PQR$.



Problem of the Week

Problem E and Solution

Good Extensions



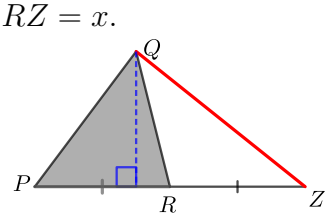
Problem

The sides of $\triangle PQR$ are extended to create $\triangle XYZ$ as follows: side RQ is extended to Y so that $RQ = QY$, side QP is extended to X so that $QP = PX$, and side PR is extended to Z so that $PR = RZ$. If the area of $\triangle XYZ = 1197$, determine the area of $\triangle PQR$.

Solution

On the above diagram, the lengths of the equal sides, $QP = PX$, $PR = RZ$, and $RQ = QY$, have been marked. Join P to Y, Q to Z, and R to X.

$\triangle PQR$ and $\triangle QRZ$ have a common altitude drawn from vertex Q to the line segment PZ.
The triangles have equal base lengths, $PR = RZ$.
Therefore, area $\triangle PQR =$ area $\triangle QRZ = x$.



At this point we can proceed to look at various other triangles with equal areas. As you go through each pair of triangles, it may help to rotate the diagram so the bases of the triangles are always on the bottom.

$\triangle PQR$ and $\triangle PQY$ have the same height and equal base lengths ($RQ = QY$).
Therefore, area $\triangle PQY =$ area $\triangle PQR = x$.

$\triangle PQY$ and $\triangle PXY$ have the same height and equal base lengths ($QP = PX$).
Therefore, area $\triangle PXY =$ area $\triangle PQY = x$.

$\triangle PXR$ and $\triangle PQR$ have the same height and equal base lengths ($QP = PX$).
Therefore, area $\triangle PXR =$ area $\triangle PQR = x$.

$\triangle PXR$ and $\triangle RXZ$ have the same height and equal base lengths ($PR = RZ$).
Therefore, area $\triangle RXZ =$ area $\triangle PXR = x$.

$\triangle QRZ$ and $\triangle QYZ$ have the same height and equal base lengths ($RQ = QY$).
Therefore, area $\triangle QYZ =$ area $\triangle QRZ = x$.

Then,

$$\begin{aligned}\text{area } \triangle XYZ &= \text{area } \triangle PXY + \text{area } \triangle PQY + \text{area } \triangle PQR + \text{area } \triangle PXR + \text{area } \triangle RXZ \\ &\qquad + \text{area } \triangle QRZ + \text{area } \triangle QYZ \\ &= x + x + x + x + x + x + x\end{aligned}$$

Therefore, $7x = 1197$ and $x = 171$.

Therefore, the area of $\triangle PQR$ is 171.

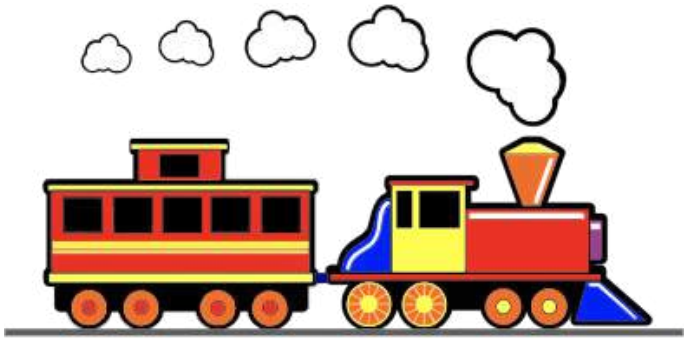
Problem of the Week

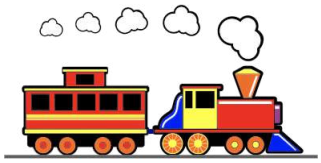
Problem E

Stay on Track

At noon, two trains are 1000 km apart. The first train is north of the second train and is traveling south. The second train is traveling east. Both trains travel at the same average speed, 100 km/h.

At what time is the **total** distance travelled by the two trains equal to the distance between the trains?





Problem of the Week

Problem E and Solution

Stay on Track

Problem

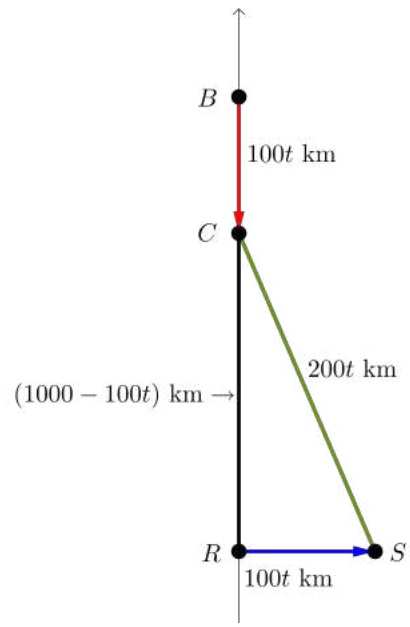
At noon, two trains are 1000 km apart. The first train is north of the second train and is traveling south. The second train is traveling east. Both trains travel at the same average speed, 100 km/h.
At what time is the **total** distance travelled by the two trains equal to the distance between the trains?

Solution

Let t be the time in hours until the total distance travelled by the two trains is equal to the distance between the trains.

Since each train is travelling at 100 km/h, the distance travelled by each train in t hours is $100t$ km. The total distance travelled by the two trains is $2 \times 100t = 200t$ km.

The diagram shows the positions of the two trains after t hours. The southbound train starts at B and moves to C. The eastbound train starts at R and moves to S. Then $BC = RS = 100t$ and since $BR = 1000$, $CR = 1000 - 100t$. We want the time t when $CS = BC + RS = 100t + 100t = 200t$.



$$\begin{aligned}
\triangle CRS \text{ is right angled, so} \qquad CS^2 &= CR^2 + RS^2 \\
(200t)^2 &= (1000 - 100t)^2 + (100t)^2 \\
40000t^2 &= 1000000 - 200000t + 10000t^2 + 10000t^2 \\
20000t^2 + 200000t - 1000000 &= 0 \\
t^2 + 10t - 50 &= 0 \\
\text{Using the Quadratic Formula,} \qquad t &= \frac{-10 \pm \sqrt{100 - 4(-50)}}{2} \\
t &= \frac{-10 \pm 10\sqrt{3}}{2} \\
t &= -5 \pm 5\sqrt{3}
\end{aligned}$$

Since $t > 0$, $-5 - 5\sqrt{3}$ is inadmissible. Therefore, $t = -5 + 5\sqrt{3} \doteq 3.66$ hours.

The distance between the two trains will be equal to their total distance travelled in $(-5 + 5\sqrt{3})$ hours, which is approximately 3 hours and 40 minutes after they leave their initial positions. The time will be approximately 3:40 pm.

Could the diagram be drawn any other way? On the next page the other two possible diagrams are briefly discussed.

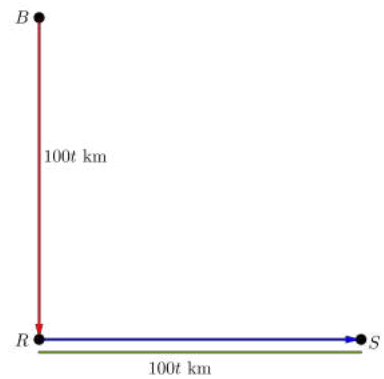
We will repeat some of the initial work and then apply it to a second diagram.

Let t be the time in hours until the total distance travelled by the two trains is equal to the distance between the trains.

Since each train is travelling at 100 km/h, the distance travelled by each train in t hours is $100t$ km. The total distance travelled by the two trains is $2 \times 100t = 200t$ km.

Maybe the train travelling south gets to a point in line with the west to east direction of the second train.

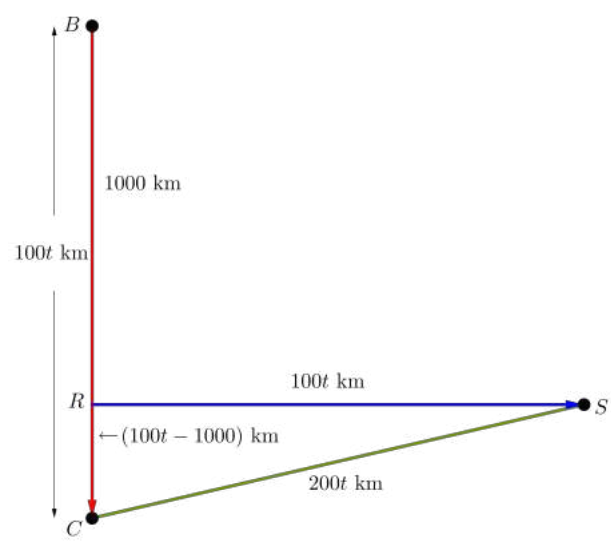
The diagram shows the positions of the two trains after t hours. Here, $BR = 1000$ km and it follows that $t = 10$ hours. The second train travels from R to S, a total of 1000 km, in the same time. But here RS is also the distance between the two trains, so is also supposed to be equal to the total distance travelled by the two trains. The total distance travelled is 2000 km, but $RS = 1000$ km. This diagram is not possible.



Is it possible that the train travelling south goes lower than the west to east line along which the second train travels?

The diagram shows the positions of the two trains after t hours. The distance travelled by the first train is represented by $BC = 100t$ km and it follows that $RC = (100t - 1000)$ km. The second train travels from R to S, a total of $100t$ km in the same time. The total distance is $CS = 200t$ km.

In any triangle, the sum of the lengths of any two sides of the triangle must be greater than the length of the third side. This is known as the triangle inequality.



In $\triangle RSC$, $RC + RS = (100t - 1000) + 100t = 200t - 1000 < 200t = CS$.

Here, the triangle inequality is not satisfied, so this "triangle" cannot represent the situation presented in this problem. It can be shown that the triangle shown on the previous page satisfies the triangle inequality.

Problem of the Week

Problem E

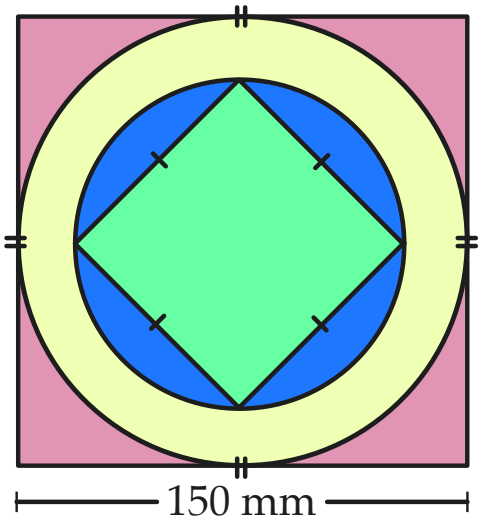
Quilting Square

Caelan is drawing a design for a quilting square. The quilting square will consist of two squares and two concentric circles.

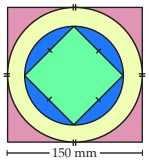
A larger square with side length 150 mm forms the boundary of the design.

Two concentric circles are drawn so that the larger of the two circles is tangent to the four sides of the larger square.

A smaller square is drawn so that the midpoint of the diagonals of the square passes through the centre of the circles and the four vertices of the square lie on the circumference of the smaller circle.



Caelan wants the area of the smaller square to be equal to the area of the region between the two circles. Determine the dimensions of the smaller square required to make the two areas equal.



Problem of the Week
Problem E and Solution
Quilting Square

Problem

Caelan is drawing a design for a quilting square. The quilting square will consist of two squares and two concentric circles. A larger square with side length 150 mm forms the boundary of the design. Two concentric circles are drawn so that the larger of the two circles is tangent to the four sides of the larger square. A smaller square is drawn so that the midpoint of the diagonals of the square passes through the centre of the circles and the four vertices of the square lie on the circumference of the smaller circle.

Caelan wants the area of the smaller square to be equal to the area of the region between the two circles. Determine the dimensions of the smaller square required to make the two areas equal.

Solution

Let x represent the side length of the smaller square.

Let D represent the diameter of the larger circle, R represent the radius of the larger circle and A_L represent the area of the larger circle.

Let d represent the diameter of the smaller circle, r represent the radius of the smaller circle and A_S represent the area of the smaller circle.

The diameter of the larger circle is equal to the side length of the quilting square. Therefore, $D = 150$ mm and $R = 75$ mm. The area of the large circle is $A_L = \pi(75)^2 = 5625\pi$ mm^2.

The length of the diagonal of the smaller square is equal to the diameter of the smaller circle. Using the Pythagorean Theorem, $d^2 = x^2 + x^2 = 2x^2$. It follows that $d = x\sqrt{2}$ m and $r = \left(\frac{x\sqrt{2}}{2}\right)$ mm.

The area of the smaller circle is $A_S = \pi\left(\frac{x\sqrt{2}}{2}\right)^2 = \frac{\pi x^2}{2}$ mm^2.

The area of the region between the two circles is $A_L - A_S = \left(5625\pi - \frac{\pi x^2}{2}\right)$ mm^2. The area of the smaller square is x^2 mm^2 and Caelan wants this equal to $A_L - A_S$.

$$\begin{aligned}
x^2 &= 5625\pi - \frac{\pi x^2}{2} \\
2x^2 &= 11\,250\pi - \pi x^2 \\
2x^2 + \pi x^2 &= 11\,250\pi \\
(2+\pi)x^2 &= 11\,250\pi \\
x^2 &= \frac{11\,250\pi}{2+\pi} \\
x &= \sqrt{\frac{11\,250\pi}{2+\pi}}, \quad x > 0 \quad \text{since side length is positive} \\
x &\approx 82.9 \text{ mm}
\end{aligned}$$

To accomplish Caelan's goal, the smaller square should have side length of approximately 83 mm.

Problem of the Week

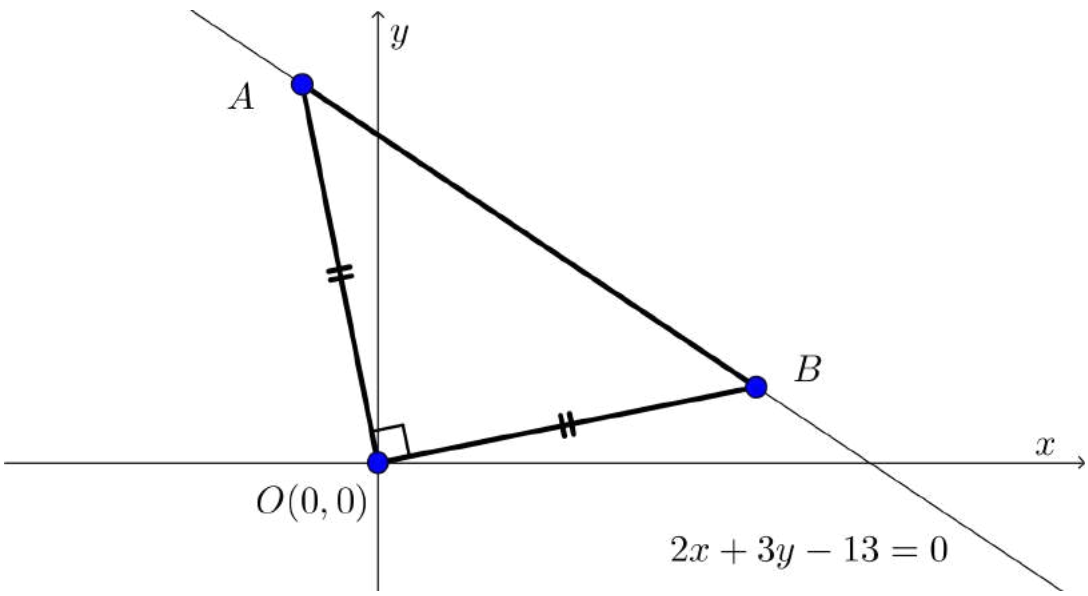
Problem E

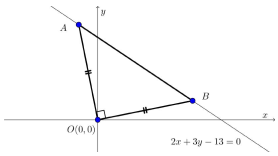
The Hypotenuse is Aligned

$\triangle OAB$ is an isosceles right-angled triangle with

- vertex O located at the origin; and
- vertices A and B located on the line $2x + 3y - 13 = 0$ such that $\angle AOB = 90^\circ$ and $OA = OB$.

Determine the area of $\triangle OAB$.





Problem of the Week

Problem E and Solution

The Hypotenuse is Aligned

Problem

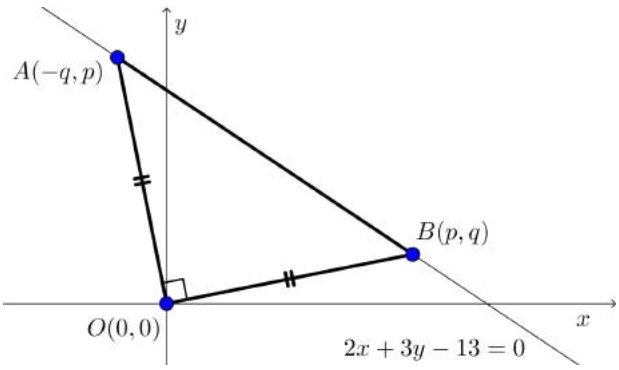
$\triangle OAB$ is an isosceles right-angled triangle with

- vertex O located at the origin; and
- vertices A and B located on the line $2x + 3y - 13 = 0$ such that $\angle AOB = 90^\circ$ and $OA = OB$.

Determine the area of $\triangle OAB$.

Solution

Solution 1



Let B have coordinates (p, q). Then the slope of $OB = \frac{q}{p}$. Since $\angle AOB = 90^\circ$, then $OB \perp OA$ and the slope of OA is the negative reciprocal of the slope of OB. Therefore, the slope of $OA = \frac{p}{-q}$. Since the triangle is isosceles with $OA = OB$, it follows that the coordinates of A are $(-q, p)$. (We can verify this by finding the length of OA and the length of OB and showing that both lengths are equal to $\sqrt{p^2 + q^2}$.)

Since $B(p, q)$ lies on the line $2x + 3y - 13 = 0$, it satisfies the equation of the line.
Therefore, $2p + 3q - 13 = 0 \quad (1)$.

Since $A(-q, p)$ lies on the line $2x + 3y - 13 = 0$, it satisfies the equation of the line.
Therefore, $-2q + 3p - 13 = 0$, or $3p - 2q - 13 = 0 \quad (2)$.

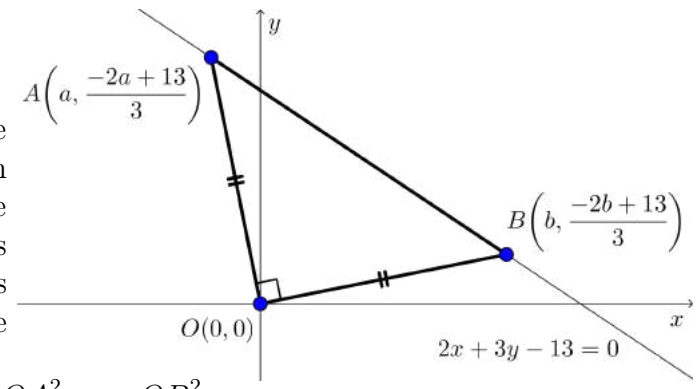
Since we have two equations and two unknowns, we can use elimination to solve for p and q.

$$\begin{aligned}
(1) \times 2: \quad 4p + 6q - 26 &= 0 \\
(2) \times 3: \quad 9p - 6q - 39 &= 0 \\
\text{Adding, we obtain}: \quad 13p - 65 &= 0 \\
p &= 5 \\
\text{Substituting in (1)}: \quad 10 + 3q - 13 &= 0 \\
3q &= 3 \\
q &= 1
\end{aligned}$$

Therefore, the point B is $(5, 1)$ and the length of OB is $\sqrt{5^2 + 1^2} = \sqrt{26}$. Since $OA = OB$, $OA = \sqrt{26}$.
$\triangle AOB$ is a right-angled triangle, so we can use OB as the base and OA as the height in the formula for the area of a triangle. Therefore, the area of $\triangle AOB$ is $\dfrac{OA \times OB}{2} = \dfrac{\sqrt{26}\sqrt{26}}{2} = 13$.

Therefore, the area of $\triangle AOB$ is 13 units^2.

Solution 2



By rearranging the given equation for the line, we obtain $y = \frac{-2x+13}{3}$. Since the points A and B are on the line, their coordinates satisfy the equation of the line. If A has x-coordinate a, then A has coordinates $\left(a, \frac{-2a+13}{3}\right)$. If B has x-coordinate b, then B has coordinates $\left(b, \frac{-2b+13}{3}\right)$. Since $\triangle OAB$ is isosceles, we know that $OA = OB$. Then

$$\begin{aligned}
OA^2 &= OB^2 \\
a^2 + \left(\frac{-2a+13}{3}\right)^2 &= b^2 + \left(\frac{-2b+13}{3}\right)^2 \\
a^2 + \frac{4a^2 - 52a + 169}{9} &= b^2 + \frac{4b^2 - 52b + 169}{9} \\
\text{Multiplying by 9 :} \quad 9a^2 + 4a^2 - 52a + 169 &= 9b^2 + 4b^2 - 52b + 169 \\
\text{Simplifying :} \quad 13a^2 - 52a + 169 &= 13b^2 - 52b + 169 \\
\text{Rearranging :} \quad 13a^2 - 13b^2 - 52a + 52b &= 0 \\
\text{Dividing by 13 :} \quad a^2 - b^2 - 4a + 4b &= 0 \\
\text{Factoring pairs :} \quad (a+b)(a-b) - 4(a-b) &= 0 \\
\text{Common factoring :} \quad (a-b)(a+b-4) &= 0
\end{aligned}$$

Solving, $a = b$ or $a = 4 - b$. Since A and B are distinct points, $a \neq b$. Therefore, $a = 4 - b$. We can rewrite $A\left(a, \frac{-2a+13}{3}\right)$ as $A\left(4-b, \frac{-2(4-b)+13}{3}\right)$ which simplifies to $A\left(4-b, \frac{2b+5}{3}\right)$.

Since $\triangle OAB$ is a right-angled triangle, we can use the Pythagorean Theorem, and $AB^2 = OA^2 + OB^2$ follows. But $OA = OB$, so this can be written $AB^2 = 2OB^2$.

$$\begin{aligned}
AB^2 &= 2OB^2 \\
(b-(4-b))^2 + \left(\frac{-2b+13}{3} - \frac{2b+5}{3}\right)^2 &= 2\left[b^2 + \left(\frac{-2b+13}{3}\right)^2\right] \\
(2b-4)^2 + \left(\frac{-4b+8}{3}\right)^2 &= 2\left[b^2 + \frac{4b^2 - 52b + 169}{9}\right] \\
4b^2 - 16b + 16 + \frac{16b^2 - 64b + 64}{9} &= 2b^2 + \frac{8b^2 - 104b + 338}{9} \\
\text{Multiplying by 9 :} \quad 36b^2 - 144b + 144 + 16b^2 - 64b + 64 &= 18b^2 + 8b^2 - 104b + 338 \\
\text{Simplifying :} \quad 52b^2 - 208b + 208 &= 26b^2 - 104b + 338 \\
\text{Rearranging :} \quad 26b^2 - 104b - 130 &= 0 \\
\text{Dividing by 26 :} \quad b^2 - 4b - 5 &= 0 \\
\text{Factoring :} \quad (b-5)(b+1) &= 0
\end{aligned}$$

It follows that $b = 5$ or $b = -1$. When $b = 5$, the point A is $(-1, 5)$ and the point B is $(5, 1)$. When $b = -1$, the point A is $(5, 1)$ and the point B is $(-1, 5)$. There are only two points. The area calculations shown in Solution 1 follow from here.

Therefore, the area of $\triangle OAB$ is 13 units^2.

Algebra (A)

TAKE ME TO THE COVER

Problem of the Week

Problem E

Worth Checking

Students in three different classes wrote the same Calculus exam. The exam was marked out of 100.

One class had 22 students in it. After writing the exam, their class average on the exam was reported as 87%. The second class had 27 students in it. After writing the exam, their class average on the exam was reported as 83%. The third class had 31 students in it. After writing the exam, their class average on the exam was reported as 81%.

Three students, Alf, Bet, and Tildi, discussed their results. Alf obtained a mark one less than Bet and Tildi obtained a mark one more than Bet.

Upon reviewing their papers, Alf and Bet both discovered addition errors on their papers. Both of their marks increased to 92. Tildi discovered that one of her questions had not been marked. This review resulted in her mark increasing to 92 as well.

These changes resulted in the exam average for all of the students in the three classes combined changing to exactly 84%.

What marks did Alf, Bet and Tildi originally have on their papers before the errors were corrected?

Problem of the Week
Problem E and Solution
Worth Checking

Problem

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Solution

Solution 1

Let Bet's original mark be b. Then Alf's original mark is $b-1$ and Tildi's original mark is $b+1$.

The total number of marks for a class can be calculated by multiplying the number of students by the class average.

The total number of marks in the first class before the error was discovered was $22 \times 87 = 1914$. The total number of marks in the second class before the error was discovered was $27 \times 83 = 2241$. The total number of marks in the third class before the error was discovered was $31 \times 81 = 2511$.

The total of the marks from the three classes before correcting the errors was $1914 + 2241 + 2511 = 6666$. To correct the errors, we subtract the three incorrect marks from the total and add the three correct marks. The new total is therefore

$$6666 - (b-1) - b - (b+1) + 3 \times 92 = 6666 - 3b + 276 = 6942 - 3b \qquad (1)$$

The total number of students in the three classes combined was $22 + 27 + 31 = 80$. Since the average after correcting the three errors was exactly 84%, the total number of marks for the three classes was $80 \times 84 = 6720 \quad (2)$.

But (1) and (2) represent the same total. Therefore, $6942 - 3b = 6720$. It follows that $3b = 222$ and $b = 74$.

Since $b = 74$, $b - 1 = 73$ and $b + 1 = 75$.

Therefore, Alf's original mark was 73, Bet's original mark was 74 and Tildi's original mark was 75.

Solution 2

Let Bet's original mark be b. Then Alf's original mark is $b-1$ and Tildi's original mark is $b+1$.

The total number of marks for a class can be calculated by multiplying the number of students by the class average.

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The total of the marks from the three classes before correcting the errors was $1914 + 2241 + 2511 = 6666$.

The total number of students in the three classes combined was $22 + 27 + 31 = 80$. Since the average after correcting the three errors was exactly 84%, the total number of marks for the three classes was $80 \times 84 = 6720$.

The total mark change is then $6720 - 6666 = 54$.

But the total mark change can also be calculated by subtracting each of their old marks from 92 and then adding the three differences.

Therefore $[92 - (b-1)] + [92 - b] + [92 - (b+1)] = 54$. Then,

$$\begin{aligned} 92 - b + 1 + 92 - b + 92 - b - 1 &= 54 \\ 276 - 3b &= 54 \\ 222 &= 3b \\ 74 &= b \end{aligned}$$

Since $b = 74$, $b - 1 = 73$ and $b + 1 = 75$.

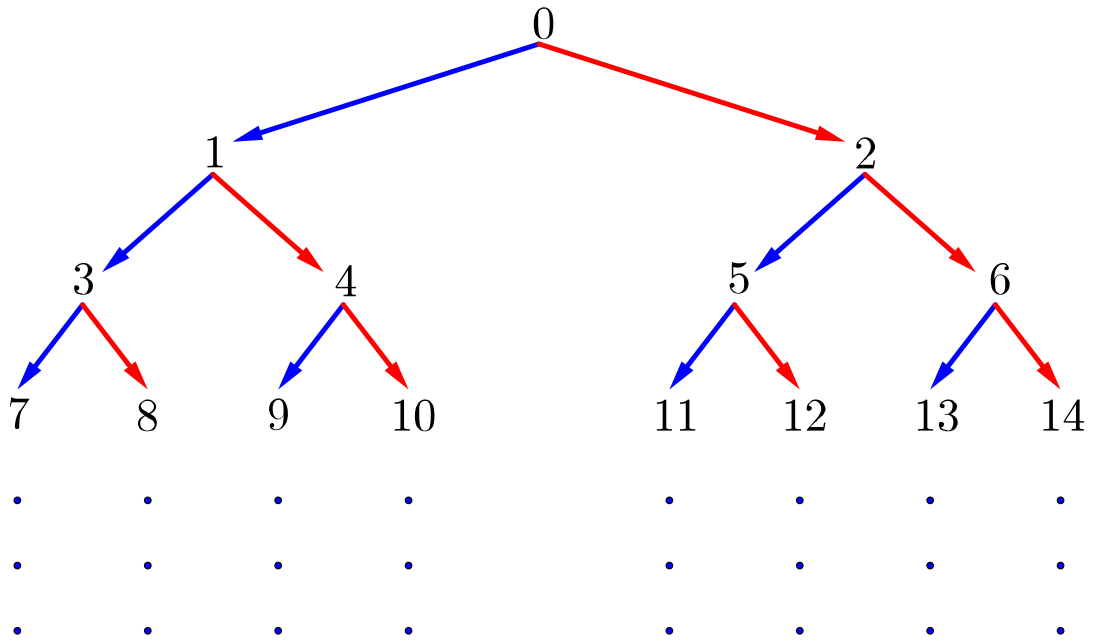
Therefore, Alf's original mark was 73, Bet's original mark was 74 and Tildi's original mark was 75.

Problem of the Week

Problem E

Find the Way

Consider the following number tree.



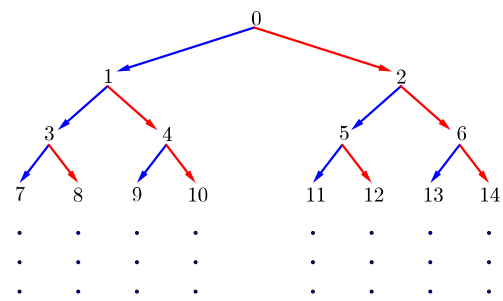
In this number tree, the integers greater than or equal to 0 are written out in increasing order, with the top row containing one integer and every row after containing twice as many integers as the row above it. Each integer is connected to two integers in the row below, one down and to the left and one down and to the right, as shown in the tree. For example, the number 5 is connected to the number 11 (down to the left) and the number 12 (down to the right) in the row below. Notice that we can get from 0 to 12 by going down right (R), down left (L), then down right (R).

What is the sequence of left (L) and right (R) movements to get from the number 0 to the number 1172 in the tree?

Problem of the Week
Problem E and Solution
Find the Way

Problem

Consider the number tree shown to the right. In this number tree, the integers greater than or equal to 0 are written out in increasing order, with the top row containing one integer and every row after containing twice as many integers as the row above it. Each integer is connected to two integers in the row below, one down and to the left and one down and to the right, as shown in the tree. For example, the number 5 is connected to the number 11 (down to the left) and the number 12 (down to the right) in the row below. Notice that we can get from 0 to 12 by going down right (R), down left (L), then down right (R).



What is the sequence of left (L) and right (R) movements to get from the number 0 to the number 1172 in the tree?

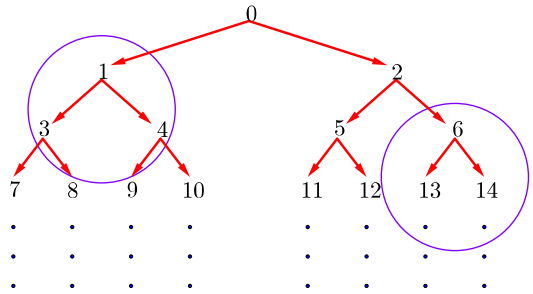
Solution

Solution 1

To begin, we will make an observation concerning the tree. When we perform a move to the left (L) from any number, we end up at an odd number. When we perform a move to the right (R) from any number, we end up at an even number. So the final move to get to the number 1172 was a move to the right (R) since 1172 is an even number. Is there a general formula which can be used to determine the number you end up at when asked to move right (R)? Is there a general formula which can be used when asked to move left (L)?

The diagram below has two parts of the tree circled. Can we discover a pattern that takes us from each initial number to the odd and even numbers below? To get from 1 to 3 we could add 2 and to get from 1 to 4 we could add 3. But doing this from 6 would not get us to 13 and 14.

As we go down the tree, each new row has twice as many numbers as the row above. Let's try multiplying the initial number by 2 and then seeing what is necessary to get to the odd and even number below. If we double 1, we get 2. Then we would need to add 1 to get to the odd number 3 below and add 2 to get to the even number 4 below. Does this work with the 6? If we double 6 and add 1, we get 13. It appears to work. If we double 6 and add 2, we get 14. It also appears to work.



So it would appear that if we make a move left (L) from any number a in the tree, the resulting number is one more than twice the value of a. That is, a move left (L) from a takes us to the number $2a + 1$ in the tree.

It would appear that if we make a move right (R) from any number a in the tree, the resulting number is two more than twice the value of a. That is, a move right (R) from a takes us to the number $2a + 2$ in the tree.

The results are true but unproven. This relationship has worked for all of the rows we have sampled but we have not proven it true in general. You will have to wait for some higher mathematics to be able to prove that this is true in general.

To go back up the tree, we could determine an inverse move to undo a move to the left (L) or to the right (R).

A move to the left (L) takes a to an odd number n such that $n = 2a + 1$. Solving this equation for a we get $a = \frac{n-1}{2}$. A move to the right (R) takes a to an even number n such that $n = 2a + 2$. Solving this equation for a we get $a = \frac{n-2}{2}$.
We can now move from 1172 up the tree to 0 using the appropriate inverse formula each time.

Initial Number	Odd or Even	Calculation	Previous Number
1172	even	$\frac{1172-2}{2}$	585
585	odd	$\frac{585-1}{2}$	292
292	even	$\frac{292-2}{2}$	145
145	odd	$\frac{145-1}{2}$	72
72	even	$\frac{72-2}{2}$	35
35	odd	$\frac{35-1}{2}$	17
17	odd	$\frac{17-1}{2}$	8
8	even	$\frac{8-2}{2}$	3
3	odd	$\frac{3-1}{2}$	1
1	odd	$\frac{1-1}{2}$	0

From our work in the table above, we can see that the path to 1172 goes through the following numbers:

$$0 \to 1 \to 3 \to 8 \to 17 \to 35 \to 72 \to 145 \to 292 \to 585 \to 1172$$

By looking at each successive number in the sequence in terms of its parity (odd or even) we can determine the required sequence of moves:

$$\begin{array}{ccccccccccc} & \text{L} & \text{L} & \text{R} & \text{L} & \text{L} & \text{R} & \text{L} & \text{R} & \text{L} & \text{R} \\ 0 \to & 1 \to & 3 \to & 8 \to & 17 \to & 35 \to & 72 \to & 145 \to & 292 \to & 585 \to & 1172 \end{array}$$

A solution that does not require the use of the unproven result is provided on the next page.

Solution 2

Which row of the tree contains the number 1172?

To get from the top integer to the rightmost integer in row 2, add 2. To get from the rightmost integer in row 2 to the rightmost integer in row 3, add 4. To get from the rightmost integer in row 3 to the rightmost integer in row 4, add 8. These numbers which are added correspond to the number of integers in the next row. Working through this we can discover that 1172 is in the 11th row.

The rightmost integer in row 10 is: $0 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 = 1022$.
The rightmost integer in row 11 is: $0 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 = 2046$.
It follows that the number 1172 is in row 11 which contains the 1024 integers ranging from 1023 (1 more than the last integer in row 10) to 2046 (the rightmost integer in row 11), inclusive.

Now we look at the commands to narrow down where we will be in the 11th row.

Row 11 can be broken into two halves, the left half contains 1023 to 1534 and the right half contains 1535 to 2046. The number 1172 is in the left half, so we needed to go left (L) first from row 1 to row 2.

Row 11 from 1023 to 1534 can be broken into two halves, the left half contains 1023 to 1278 and the right half contains 1279 to 1534. The number 1172 is in the left half, so we needed to go left (L) from row 2 to row 3.

Row 11 from 1023 to 1278 can be broken into two halves, the left half contains 1023 to 1150 and the right half contains 1151 to 1278. The number 1172 is in the right half, so we needed to go right (R) from row 3 to row 4.

Row 11 from 1151 to 1278 can be broken into two halves, the left half contains 1151 to 1214 and the right half contains 1215 to 1278. The number 1172 is in the left half, so we needed to go left (L) from row 4 to row 5.

Row 11 from 1151 to 1214 can be broken into two halves, the left half contains 1151 to 1182 and the right half contains 1183 to 1214. The number 1172 is in the left half, so we needed to go left (L) from row 5 to row 6.

Row 11 from 1151 to 1182 can be broken into two halves, the left half contains 1151 to 1166 and the right half contains 1167 to 1182. The number 1172 is in the right half, so we needed to go right (R) from row 6 to row 7.

Row 11 from 1167 to 1182 can be broken into two halves, the left half contains 1167 to 1174 and the right half contains 1175 to 1182. The number 1172 is in the left half, so we needed to go left (L) from row 7 to row 8.

Row 11 from 1167 to 1174 can be broken into two halves, the left half contains 1167 to 1170 and the right half contains 1171 to 1174. The number 1172 is in the right half, so we needed to go right (R) from row 8 to row 9.

Row 11 from 1171 to 1174 can be broken into two halves, the left half contains 1171 to 1172 and the right half contains 1173 to 1174. The number 1172 is in the left half, so we needed to go left (L) from row 9 to row 10.

And finally, row 11 from 1171 to 1172 can be broken into two halves, the left half contains 1171 and the right half contains 1172. The number 1172 is in the right half, so we needed to go right (R) from row 10 to row 11.

Therefore, we get to 1172 using the following sequence of movements:

$$\mathrm{L} \to \mathrm{L} \to \mathrm{R} \to \mathrm{L} \to \mathrm{L} \to \mathrm{R} \to \mathrm{L} \to \mathrm{R} \to \mathrm{L} \to \mathrm{R}$$

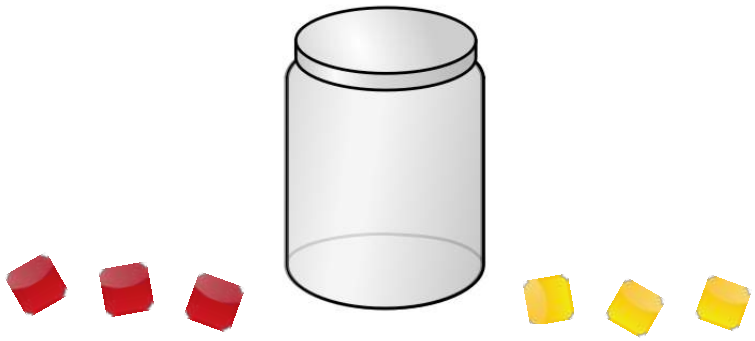
Problem of the Week

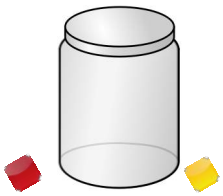
Problem E

Candies Anyone?

A jar contains only small red and small yellow candies. Another 30 red candies are added to the candies already in the jar so that one-third of the total number of candies in the jar are red candies. At this point, 30 yellow candies are added to the jar, and now three-tenths of the total number of candies in the jar are red candies.

What fraction of the number of the candies originally in the jar were red candies?





Problem of the Week

Problem E and Solution

Candies Anyone?

Problem

A jar contains only small red and small yellow candies. Another 30 red candies are added to the candies already in the jar so that one-third of the total number of candies in the jar are red candies. At this point, 30 yellow candies are added to the jar, and now three-tenths of the total number of candies in the jar are red candies. What fraction of the number of the candies originally in the jar were red candies?

Solution

Let r represent the number of red candies originally in the jar.
Let y represent the number of yellow candies originally in the jar.
Then $r + y$ represents the total number of candies originally in the jar.

After adding 30 red candies to the candies already in the jar, there are $(r + 30)$ red candies in the jar and a total of $(r + y + 30)$ candies in the jar. Now one-third of the candies in the jar are red candies, so

$$\begin{aligned} \frac{r+30}{r+y+30} &= \frac{1}{3} \\ 3(r+30) &= 1(r+y+30) \\ 3r+90 &= r+y+30 \\ 2r+60 &= y \qquad (1) \end{aligned}$$

After adding 30 yellow candies to the candies in the jar, there are $(r + 30)$ red candies in the jar and a total of $(r + y + 60)$ candies in the jar. Now three-tenths of the candies in the jar are red candies, so

$$\begin{aligned} \frac{r+30}{r+y+60} &= \frac{3}{10} \\ 10(r+30) &= 3(r+y+60) \\ 10r+300 &= 3r+3y+180 \\ 7r-3y &= -120 \qquad (2) \end{aligned}$$

Substituting (1) into (2),

$$\begin{aligned} 7r-3(2r+60) &= -120 \\ 7r-6r-180 &= -120 \\ r &= 60 \end{aligned}$$

Substituting $r = 60$ into (1), we get $y = 180$.

Therefore, there were originally 60 red and 180 yellow candies in the jar, and $\frac{60}{60+180} = \frac{60}{240} = \frac{1}{4}$ of the candies originally in the jar were red.

Problem of the Week

Problem E

The Area of the Year

In the diagram, $\triangle AB_1C_1$ is right-angled with $AB_1 = 2$ and $AC_1 = 5$. Lines AB_1 and AC_1 are extended and many more points are labelled at intervals of 1 unit, so that

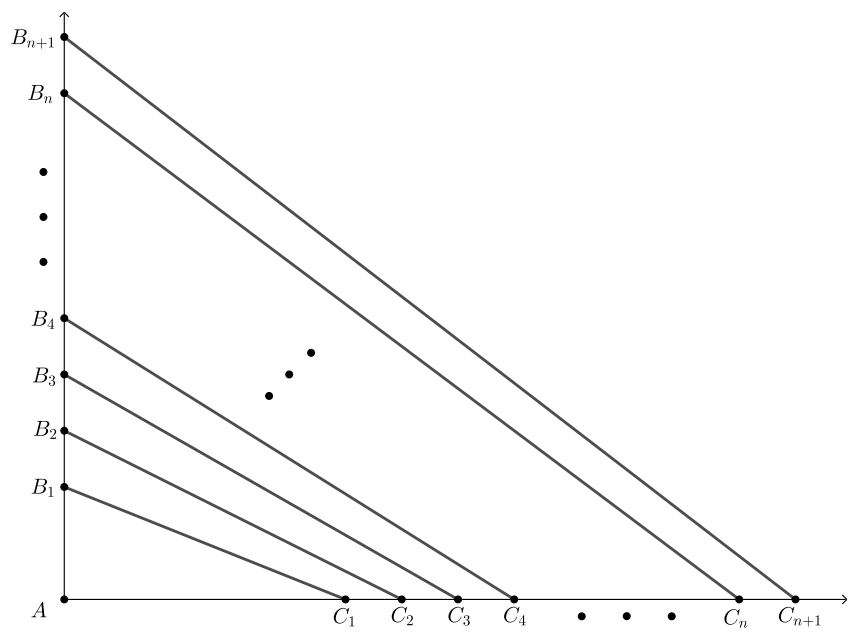
$$B_1B_2 = B_2B_3 = B_3B_4 = B_4B_5 = \cdots = 1, \text{ and}$$

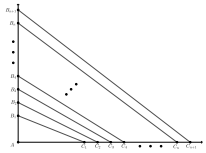
$$C_1C_2 = C_2C_3 = C_3C_4 = C_4C_5 = \cdots = 1.$$

In fact, $B_1B_j = j - 1$ and $C_1C_k = k - 1$ for any positive integers j and k.

For example, $B_1B_5 = 5 - 1 = 4$ and $C_1C_4 = 4 - 1 = 3$.

Determine the value of n so that the area of quadrilateral $B_nB_{n+1}C_{n+1}C_n$ is 2020. That is, determine the value of n so that the area of the quadrilateral with vertices B_n, B_{n+1}, C_{n+1}, and C_n is 2020.



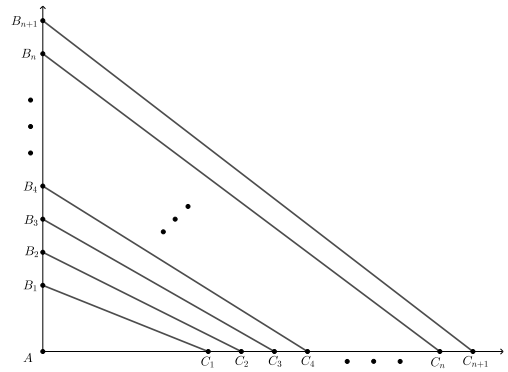


Problem of the Week

Problem E and Solution

The Area of the Year

Problem



In the diagram, $\triangle AB_1C_1$ is right-angled with $AB_1 = 2$ and $AC_1 = 5$. Lines AB_1 and AC_1 are extended and many more points are labelled at intervals of 1 unit, so that $B_1B_2 = B_2B_3 = B_3B_4 = B_4B_5 = \cdots = 1$, and $C_1C_2 = C_2C_3 = C_3C_4 = C_4C_5 = \cdots = 1$. In fact, $B_1B_j = j - 1$ and $C_1C_k = k - 1$ for any positive integers j and k. For example, $B_1B_5 = 5 - 1 = 4$ and $C_1C_4 = 4 - 1 = 3$. Determine the value of n so that the area of quadrilateral $B_nB_{n+1}C_{n+1}C_n$ is 2020.

Solution

Solution 1

In order to solve the problem, looking at the calculation of a specific area may prove helpful.

So let's determine the area of quadrilateral $B_4B_5C_5C_4$.

$$\begin{aligned}
\text{Area of quadrilateral } B_4B_5C_5C_4 &= \text{Area } \triangle B_5AC_5 - \text{Area } \triangle B_4AC_4 \\
&= \frac{1}{2}(AB_5)(AC_5) - \frac{1}{2}(AB_4)(AC_4) \\
&= \frac{1}{2}(2 + (5-1))(5 + (5-1)) - \frac{1}{2}(2 + (4-1))(5 + (4-1)) \\
&= \frac{1}{2}(6)(9) - \frac{1}{2}(5)(8) \\
&= 27 - 20 \\
&= 7 \text{ units}^2
\end{aligned}$$

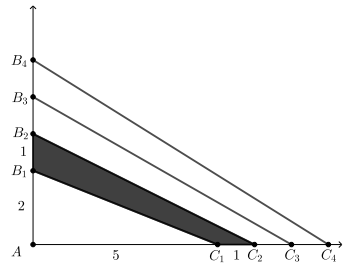
We will solve the problem by following what we did in the above example.

$$\begin{aligned}
\text{Area of quad. } B_nB_{n+1}C_{n+1}C_n &= \text{Area } \triangle B_{n+1}AC_{n+1} - \text{Area } \triangle B_nAC_n \\
2020 &= \frac{1}{2}(AB_{n+1})(AC_{n+1}) - \frac{1}{2}(AB_n)(AC_n) \\
2020 &= \frac{1}{2}(2 + ((n+1)-1))(5 + ((n+1)-1)) - \frac{1}{2}(2 + (n-1))(5 + (n-1)) \\
2020 &= \frac{1}{2}(2+n)(5+n) - \frac{1}{2}(1+n)(4+n) \\
4040 &= (2+n)(5+n) - (1+n)(4+n) \qquad \text{multiplying by 2} \\
4040 &= n^2 + 7n + 10 - (n^2 + 5n + 4) \\
4040 &= n^2 + 7n + 10 - n^2 - 5n - 4 \\
4040 &= 2n + 6 \\
4034 &= 2n \\
2017 &= n
\end{aligned}$$

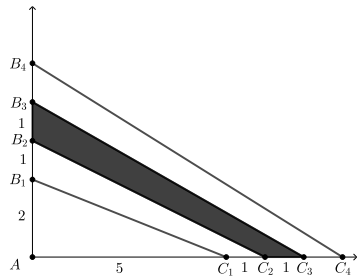
Therefore, the value of n is 2017.

Solution 2

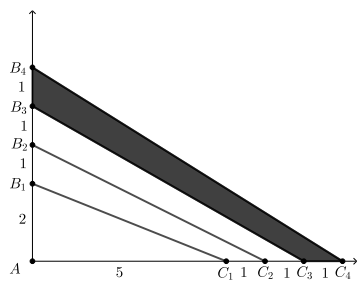
In this solution we look for a pattern in the area calculations.



$$\begin{aligned}
\text{Area of quad. } B_1B_2C_2C_1 &= \text{Area } \triangle B_2AC_2 - \text{Area } \triangle B_1AC_1 \\
&= \frac{1}{2}(AB_2)(AC_2) - \frac{1}{2}(AB_1)(AC_1) \\
&= \frac{1}{2}(3)(6) - \frac{1}{2}(2)(5) \\
&= 9 - 5 \\
\text{Area of first quad.} &= 4 \text{ units}^2
\end{aligned}$$



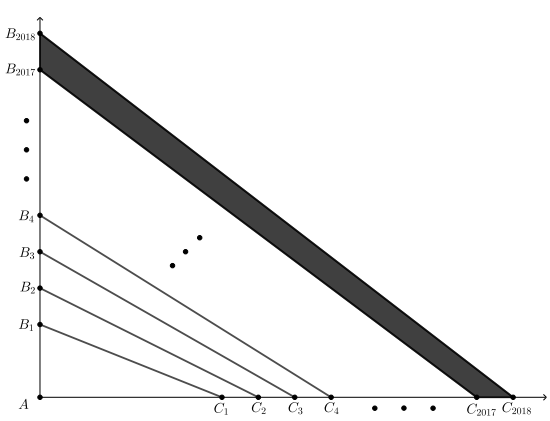
$$\begin{aligned}
\text{Area of quad. } B_2B_3C_3C_2 &= \text{Area } \triangle B_3AC_3 - \text{Area } \triangle B_2AC_2 \\
&= \frac{1}{2}(AB_3)(AC_3) - \frac{1}{2}(AB_2)(AC_2) \\
&= \frac{1}{2}(4)(7) - \frac{1}{2}(3)(6) \\
&= 14 - 9 \\
\text{Area of second quad.} &= 5 \text{ units}^2
\end{aligned}$$



$$\begin{aligned}
\text{Area of quad. } B_3B_4C_4C_3 &= \text{Area } \triangle B_4AC_4 - \text{Area } \triangle B_3AC_3 \\
&= \frac{1}{2}(AB_4)(AC_4) - \frac{1}{2}(AB_3)(AC_3) \\
&= \frac{1}{2}(5)(8) - \frac{1}{2}(4)(7) \\
&= 20 - 14 \\
\text{Area of third quad.} &= 6 \text{ units}^2
\end{aligned}$$

A possible pattern has emerged. The area of the quadrilateral appears to be three more than the position of the quadrilateral on the stack of consecutive quadrilaterals. If we want the area to be 2020, then it should be the 2017th quadrilateral. That is, it should be the quadrilateral with vertices $B_{2017}B_{2018}C_{2018}C_{2017}$. Therefore, the value of n is 2017.

We can verify this value of n using the area calculation. Recall from the problem statement, $B_1B_j = j - 1$ and $C_1C_k = k - 1$ for any positive integers j and k.
So, $B_1B_{2017} = 2017 - 1 = 2016$. Then $AB_{2017} = AB_1 + B_1B_{2017} = 2 + 2016 = 2018$.
Since $AB_{2018} = AB_{2017} + 1$, it follows that $AB_{2018} = 2019$.
Also, $C_1C_{2017} = 2017 - 1 = 2016$. Then $AC_{2017} = AC_1 + C_1C_{2017} = 5 + 2016 = 2021$.
Since $AC_{2018} = AC_{2017} + 1$, it follows that $AC_{2018} = 2022$.



$$\begin{aligned}
&\text{Area of quadrilateral } B_{2017}B_{2018}C_{2018}C_{2017} \\
&= \text{Area } \triangle B_{2018}AC_{2018} - \text{Area } \triangle B_{2017}AC_{2017} \\
&= \frac{1}{2}(AB_{2018})(AC_{2018}) - \frac{1}{2}(AB_{2017})(AC_{2017}) \\
&= \frac{1}{2}(2019)(2022) - \frac{1}{2}(2018)(2021) \\
&= 2\,041\,209 - 2\,039\,189 \\
&= 2020 \text{ units}^2
\end{aligned}$$

Problem of the Week

Problem E

What are the Possibilities?

You may be surprised to learn that the equation $(x^2 - 5x + 5)^{x^2+4x-60} = 1$ has five solutions.

Determine all five values of x that satisfy the equation.

Problem of the Week
Problem E and Solution
What are the Possibilities?

Problem
Determine all values of x that satisfy the equation $(x^2 - 5x + 5)^{x^2+4x-60} = 1$.

Solution

Let's consider the ways that an expression of the form a^b can be 1:

- The base, a, is 1.

  In this case, the exponent can be any value and we need to solve $x^2 - 5x + 5 = 1$.

  $$\begin{aligned} x^2 - 5x + 5 &= 1 \\ x^2 - 5x + 4 &= 0 \\ (x-4)(x-1) &= 0 \end{aligned}$$

  So $x = 4$ or $x = 1$.

- The exponent, b, is 0.

  In this case, the base can be any number other than 0 and we need to solve $x^2 + 4x - 60 = 0$.

  $$\begin{aligned} x^2 + 4x - 60 &= 0 \\ (x-6)(x+10) &= 0 \end{aligned}$$

  So $x = 6$ or $x = -10$.

  When $x = 6$, the base is $6^2 - 5(6) + 5 = 11 \neq 0$. That is, when $x = 6$, the base does not equal 0.
  When $x = -10$, the base is $(-10)^2 - 5(-10) + 5 = 155 \neq 0$. That is, when $x = -10$, the base does not equal 0.

- The base, a, is -1 and the exponent, b, is even.

  We first need to solve $x^2 - 5x + 5 = -1$.

  $$\begin{aligned} x^2 - 5x + 5 &= -1 \\ x^2 - 5x + 6 &= 0 \\ (x-2)(x-3) &= 0 \end{aligned}$$

  So $x = 2$ or $x = 3$.

  When $x = 2$, the exponent is $2^2 + 4(2) - 60 = -48$, which is even.
  Therefore, when $x = 2$, $(x^2 - 5x + 5)^{x^2+4x-60} = 1$.

  When $x = 3$, the exponent is $3^2 + 4(3) - 60 = -39$, which is odd.
  Therefore, when $x = 3$, $(x^2 - 5x + 5)^{x^2+4x-60} = -1$. So $x = 3$ is not a solution.

Therefore, the values of x that satisfy $(x^2 - 5x + 5)^{x^2+4x-60} = 1$ are $x = -10$, $x = 1$, $x = 2$, $x = 4$ and $x = 6$. There are five values of x which satisfy the equation.

Problem of the Week

Problem E

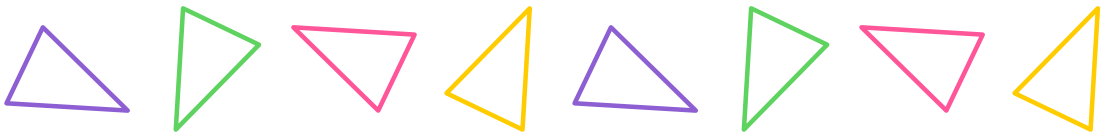
Terry's Triangles

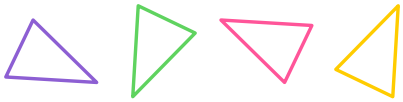
Terry is drawing isosceles triangles with side lengths a, b, and c such that

$$a = y - x$$
$$b = x + z$$
$$c = y - z$$

Where x, y, and z are positive integers and $x + y + z < 10$.

Find all the possible triples (a, b, c) that satisfy this.





Problem of the Week
Problem E and Solution
Terry's Triangles

Problem

Terry is drawing isosceles triangles with side lengths a, b, and c such that

$$a = y - x$$
$$b = x + z$$
$$c = y - z$$

Where x, y, and z are positive integers and $x + y + z < 10$.

Find all the possible triples (a, b, c) that satisfy this.

Solution

In an isosceles triangle, two sides must have equal length. So we need to consider three cases: $a = b$, $b = c$, and $a = c$. Also, in order for a, b, and c to represent side lengths of a triangle, they must be positive numbers and the sum of any two side lengths must be greater than the other side length.

Case 1: $a = b$

If $a = b$, then $y - x = x + z$, so $y = 2x + z$. We can make a table of all the values of x, y, and z that satisfy this equation as well as $x + y + z < 10$, and then find the corresponding values of a, b, and c and check if they are valid side lengths.

x	y	z	a	b	c	Valid?
1	3	1	2	2	2	Yes
1	4	2	3	3	2	Yes
1	5	3	4	4	2	Yes
2	5	1	3	3	4	Yes

Case 2: $b = c$

If $b = c$, then $x + z = y - z$, so $y = x + 2z$. As in Case 1, we can write the possible values of x, y, z, a, b, and c in a table.

x	y	z	a	b	c	Valid?
1	3	1	2	2	2	Yes
2	4	1	2	3	3	Yes
3	5	1	2	4	4	Yes
1	5	2	4	3	3	Yes

Case 3: $a = c$

If $a = c$, then $y - x = y - z$, so $x = z$. As in previous cases, we can write the possible values of x, y, z, a, b, and c in a table.

x	y	z	a	b	c	Valid?
1	1	1	0	2	0	No (a and c are not positive)
1	2	1	1	2	1	No ($a + c \not> b$)
1	3	1	2	2	2	Yes
1	4	1	3	2	3	Yes
1	5	1	4	2	4	Yes
1	6	1	5	2	5	Yes
1	7	1	6	2	6	Yes
2	1	2	-1	4	-1	No (a and c are not positive)
2	2	2	0	4	0	No (a and c are not positive)
2	3	2	1	4	1	No ($a + c \not> b$)
2	4	2	2	4	2	No ($a + c \not> b$)
2	5	2	3	4	3	Yes
3	1	3	-2	6	-2	No (a and c are not positive)
3	2	3	-1	6	-1	No (a and c are not positive)
3	3	3	0	6	0	No (a and c are not positive)
4	1	4	-3	8	-3	No (a and c are not positive)

Therefore, there are 12 possible triples (a, b, c). They are listed below.

$$(2,2,2) \quad (3,3,2) \quad (4,4,2) \quad (3,3,4)$$
$$(2,3,3) \quad (2,4,4) \quad (4,3,3)$$
$$(3,2,3) \quad (4,2,4) \quad (5,2,5) \quad (6,2,6) \quad (3,4,3)$$

Problem of the Week

Problem E

What Exponent?

A two-digit positive integer x exists such that when the expression $(10^x - x)$ is evaluated, the sum of the digits of the difference is 300.

Determine the value of the exponent x.

$$[10^x - x]$$

$[10^x - x]$

Problem of the Week
Problem E and Solution
What Exponent?

Problem

A two-digit positive integer x exists such that when the expression $(10^x - x)$ is evaluated, the sum of the digits of the difference is 300.

Determine the value of the exponent x.

Solution

When 10^x is written out there is a one followed by x zeroes, a total of $(x + 1)$ digits. When we attempt to subtract x from this number using the traditional algorithm for subtraction we are required to borrow.

So instead, we will convert to a question that has the same answer and will be easier to work with. The expression $(10^x - x)$ will have the same difference as $(10^x - 1) - (x - 1)$. The number $(10^x - 1)$ is one less than 10^x. This new number is then made up of x nines. Let's look at a few specific cases to see if we can determine what is happening in this problem.

x	$10^x - 1$	$x - 1$	$(10^x - 1) - (x - 1)$
10	9 999 999 999	9	9 999 999 990
11	99 999 999 999	10	99 999 999 989
12	999 999 999 999	11	999 999 999 988
13	9 999 999 999 999	12	9 999 999 999 987

A pattern seems to be forming in the digits of the difference. As the exponent x increases by 1, the number of nines to the left of the rightmost two digits increases by 1. If we ignore the rightmost two digits, we can predict the number of leading nines in the difference by comparing to the exponent.

Exponent	Number of nines to the left of two rightmost digits
10	8
11	9
12	10
13	11
x	$x - 2$

We have not proven this result. We will use the result to find the value of x that gives the difference a digit sum of 300. We will then verify that the result is correct.

Ignoring the rightmost two digits, the difference would be a number made up of $(x-2)$ nines and the digit sum would be $9(x-2)$. We want x so that

$$\begin{aligned} 9(x-2) &< 300 \\ 9x-18 &< 300 \\ 9x &< 318 \\ x &< 35\frac{1}{3} \end{aligned}$$

Since the exponent x is a positive two-digit integer, $x \leq 35$. Note, this is not necessarily the answer. It simply is the maximum value that x could be.

If $x = 35$, the rightmost two digits are $100 - 35 = 65$. The digit sum is $33 \times 9 + 6 + 5 = 308$. This is not the digit sum we are looking for.

If $x = 34$, the rightmost two digits are $100 - 34 = 66$. The digit sum is $32 \times 9 + 6 + 6 = 300$.

If our pattern is correct, the value of x is 34. We will verify the result. Remember,

$$10^{34} - 34 = (10^{34} - 1) - (34 - 1) = (10^{34} - 1) - 33$$

$(10^{34} - 1)$ is a 34-digit number consisting of exactly 34 nines.

$$\begin{array}{cccccccccccc} 9 & 9 & 9 & 9 & \cdots & 9 & 9 & 9 & 9 & 9 & 9 \\ & & & & & & & & - & 3 & 3 \\ \hline 9 & 9 & 9 & 9 & \cdots & 9 & 9 & 9 & 9 & 6 & 6 \end{array}$$

The difference still has 34 digits. The total number of digits is made up of the rightmost two digits and 32 nines. The digit sum is $6 + 6 + 32 \times 9 = 300$.

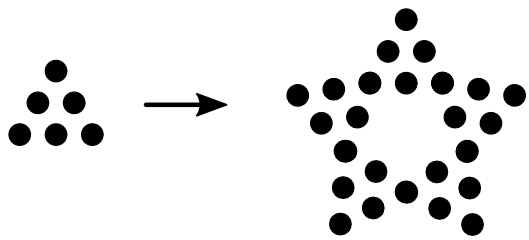
Our result is confirmed. When $x = 34$, the sum of the digits of the difference $10^{34} - 34$ is 300.

Problem of the Week

Problem E

Dot Dot Dot

Ponto forms a triangle using dots, and then puts five of these triangles together to make a star. When he does this, the dots in the bottom corners of each adjacent triangle overlap. An example is shown below.

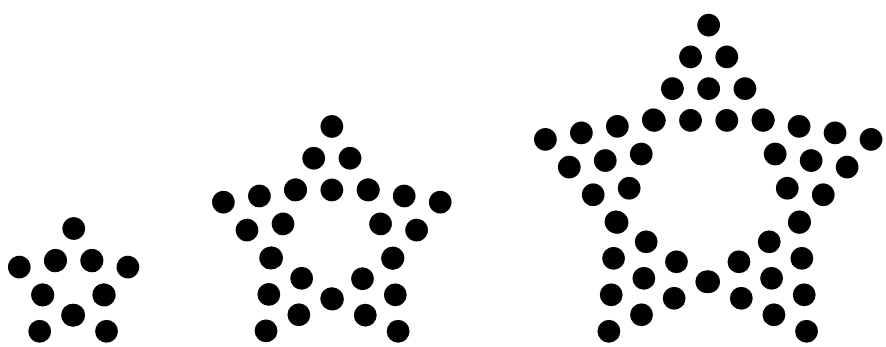


Ponto creates a sequence of such stars as follows.

- Each triangle in the first star has a row with one dot on top of a row with two dots.
- Each triangle in all subsequent stars has one more row than the triangles in the previous star. This new row is placed at the bottom of each triangle and has one more dot than the row above it.

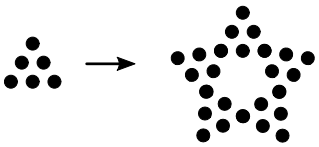
The first three stars in the sequence are shown. They have 10, 25, and 45 dots showing, respectively.

Which star in the sequence will have 20 020 dots showing?



NOTE:
In solving the above problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$



Problem of the Week

Problem E and Solution

Dot Dot Dot

Problem

Ponto forms a triangle using dots, and then puts five of these triangles together to make a star. When he does this, the dots in the bottom corners of each adjacent triangle overlap. An example is shown above. Ponto creates a sequence of such stars as follows.

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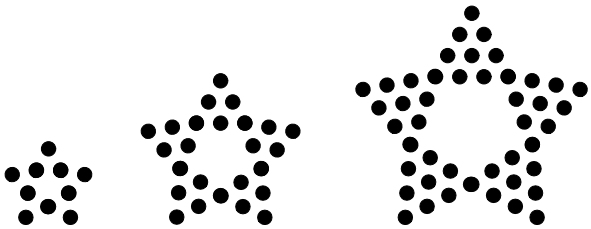
The first three stars in the sequence are shown. They have 10, 25, and 45 dots showing, respectively. Which star in the sequence will have 20 020 dots showing?

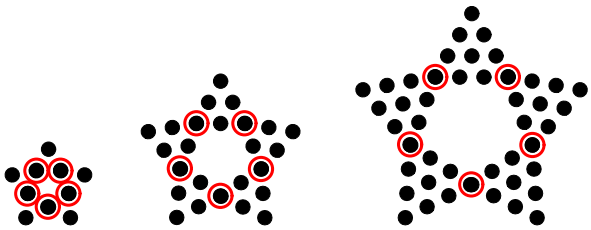


Solution

We will write an equation to represent the number of dots in each star in the sequence. We know that each star is made of 5 triangles that overlap at the corners. Thus, the total number of dots in each star is equal to

$$(\text{number of dots in each triangle}) \times 5 - (\text{number of overlapped dots})$$

The overlapped dots are circled in the diagram below. They are the dots in the bottom corners of each triangle.



We can see that each star in the sequence will have 5 overlapped dots. So we can simplify our expression for the total number of dots to

$$(\text{number of dots in each triangle}) \times 5 - 5$$

Now let's look at how many dots are in each triangle.

- The first star has $1 + 2 = 3$ dots in each triangle.
- The second star has $1 + 2 + 3 = 6$ dots in each triangle.
- The third star has $1 + 2 + 3 + 4 = 10$ dots in each triangle.

Following this pattern, we can see that the fourth star would have $1 + 2 + 3 + 4 + 5 = 15$ dots in each triangle. Thus, the nth star would have $1 + 2 + \cdots + n + (n + 1)$ dots in each triangle.

We know the sum of the first n positive integers is $\dfrac{n(n+1)}{2}$. Thus, the sum of the first $(n + 1)$ positive integers is $\dfrac{(n+1)(n+2)}{2}$. So the nth star would have $\dfrac{(n+1)(n+2)}{2}$ dots in each triangle. If we let t_n represent the number of dots in the nth star in the sequence, then we can write the following equation.

$$t_n = \frac{(n+1)(n+2)}{2} \times 5 - 5$$

We want to know which star has 20 020 dots. This means we want to find the value of n when $t_n = 20\,020$.

$$\begin{aligned}
20020 &= \frac{(n+1)(n+2)}{2} \times 5 - 5 \\
20025 &= \frac{(n+1)(n+2)}{2} \times 5 \\
\frac{20025}{5} &= \frac{(n+1)(n+2)}{2} \\
4005 \times 2 &= (n+1)(n+2) \\
8010 &= n^2 + 3n + 2 \\
0 &= n^2 + 3n - 8008 \\
0 &= (n - 88)(n + 91)
\end{aligned}$$

Thus $n = 88$, or $n = -91$. Since n must be a positive integer, it follows that $n = 88$.

Therefore, the 88th star has 20 020 dots.

Extension: Show that for any positive integer n, we in fact have

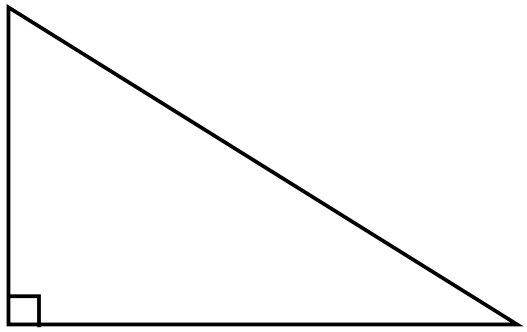
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

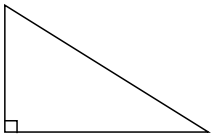
Problem of the Week

Problem E

That's Right

The side lengths of a right-angled triangle are all positive two-digit integers. If the digits representing the length of the hypotenuse are the digits of one of the side lengths written in the reverse order, find all the possible lengths of the hypotenuse.





Problem of the Week

Problem E and Solution

That's Right

Problem

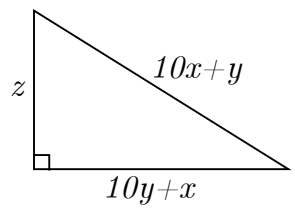
The side lengths of a right-angled triangle are all positive two-digit integers. If the digits representing the length of the hypotenuse are the digits of one of the side lengths written in the reverse order, find all the possible lengths of the hypotenuse.

Solution

Let x represent the tens digit of the hypotenuse such that x is an integer from 1 to 9. Let y represent the units digit of the hypotenuse such that y is an integer from 1 to 9. Then the length of the hypotenuse is $10x + y$.

Since one of the sides has the same digits as the hypotenuse in reverse order, the length of this side is $10y + x$.

Let the third side be z such that z is a two digit integer.



$$\begin{aligned}
\text{Using the Pythagorean Theorem:} \quad (10y+x)^2 + z^2 &= (10x+y)^2 \\
\text{Expanding:} \quad 100y^2 + 20xy + x^2 + z^2 &= 100x^2 + 20xy + y^2 \\
\text{Rearranging:} \quad z^2 &= 99x^2 - 99y^2 \\
\text{Factoring:} \quad z^2 &= 99(x-y)(x+y)
\end{aligned}$$

Since z^2 is a perfect square, $99(x+y)(x-y)$ must also be a perfect square.
But $99(x+y)(x-y) = 9(11)(x+y)(x-y)$. So, to be a perfect square, $(x+y)(x-y)$ must contain a factor of 11. Since x and y are each integers from 1 to 9, $x - y$ cannot be equal to 11 and $x + y$ cannot be greater than 18, and so we must have $x + y = 11$. Now, for $9(11)(x+y)(x-y) = 9(11)(11)(x-y)$ to be a perfect square, $x - y$ must be a perfect square.
Since x and y are integers from 1 to 9, there are three possibilities for $x - y$ that give a perfect square; $x - y = 1$, $x - y = 4$ or $x - y = 9$. These are the three possibilities:
$(x+y)(x-y) = 11(1)$ or $(x+y)(x-y) = 11(4)$ or $(x+y)(x-y) = 11(9)$.

Case 1: $(x+y)(x-y) = 11(1)$
If $x + y = 11$ and $x - y = 1$, we solve the system of equations obtaining $x = 6$ and $y = 5$. This gives a hypotenuse of $10x + y = 65$ and second side $10y + x = 56$. Then solving for z, $z^2 = 99(x+y)(x-y) = 99(11)(1) = 1089$ and $z = 33$. This solution is easily confirmed but is it the only solution?

Case 2: $(x+y)(x-y) = 11(4)$
If $x + y = 11$ and $x - y = 4$, we solve the system of equations obtaining $x = 7.5$ and $y = 3.5$. But x and y are not integers so this solution is inadmissible.

Case 3: $(x+y)(x-y) = 11(9)$
If $x + y = 11$ and $x - y = 9$, we solve the system of equations obtaining $x = 10$ and $y = 1$. But x must be an integer from 1 to 9 so this solution is inadmissible.

Therefore, the only solution is a hypotenuse of length 65.

Problem of the Week

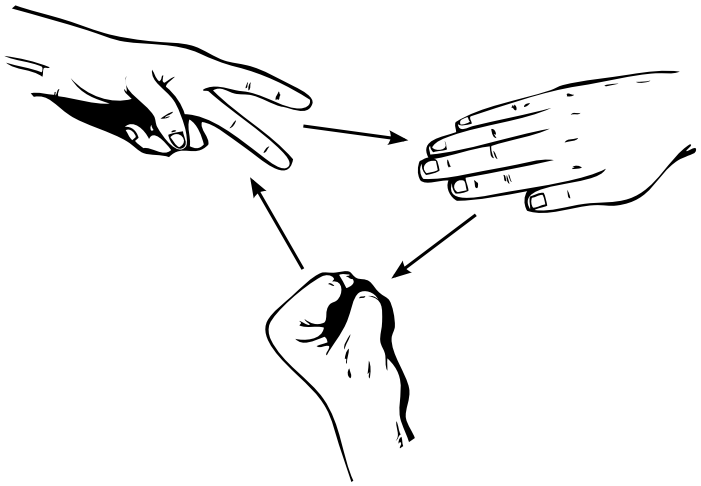
Problem E

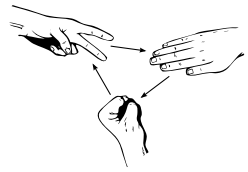
Rock Out

Sameer likes playing rock paper scissors so much that he started a club at his school. At every club meeting, each person plays rock paper scissors against each of the other people at the meeting the same number of times.

There were 40% more people at the second club meeting than the first. The total number of rock paper scissors matches played at the second meeting was twice the number of matches played at the first meeting. The number of matches each person plays against each of the other people at the meeting stays the same from meeting to meeting.

How many people were at the first meeting?





Problem of the Week
Problem E and Solution
Rock Out

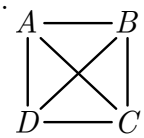
Problem

Sameer likes playing rock paper scissors so much that he started a club at his school. At every club meeting, each person plays rock paper scissors against each of the other people at the meeting the same number of times. There were 40% more people at the second club meeting than the first. The total number of rock paper scissors matches played at the second meeting was twice the number of matches played at the first meeting. The number of matches each person plays against each of the other people at the meeting stays the same from meeting to meeting. How many people were at the first meeting?

Solution

Let p represent the number of people at the first meeting. Then the number of people at the second meeting is $1.4p$. Note that both p and $1.4p$ must be positive integers.

We need to first establish how many games are played. Suppose there were 4 people ($A, B, C,$ and D), and each person played against every other person once, as shown to the right.



We can see there would be 6 matches played ($AB, AC, AD, BC, BD,$ and CD). This is easy to see in a small diagram, but as the number of people and matches increase, the diagrams become hard to read, so we need to find a general solution.

Often, when counting something like this we "double-count" by mistake. We think that because there are 4 people and each person plays the 3 other people, that means there are $4 \times 3 = 12$ matches in total. However, we have counted each match twice, so we need to divide the result by 2. So if there were 4 people, with each person playing every other person once, there would be $\frac{4\times 3}{2} = 6$ matches played in total.

In general, if there are p people and each person plays against every other person once, there would be $\frac{p(p-1)}{2}$ matches played in total. If each person plays against every other person n times, the total number of matches would need to be multiplied by n, to obtain $n\left(\frac{p(p-1)}{2}\right)$. This represents the total number of matches at the first meeting. Now in the second meeting, the number of people is $1.4p$, so the total number of matches in the second meeting would be $n\left(\frac{1.4p(1.4p-1)}{2}\right)$.

We know that the number of matches played at the second meeting is twice the number of matches played at the first meeting. So,

$$n\left(\frac{1.4p(1.4p-1)}{2}\right) = 2\left[n\left(\frac{p(p-1)}{2}\right)\right]$$

Dividing both sides by $\frac{n}{2}$, $n \neq 0$, this simplifies to

$$1.4p(1.4p-1) = 2p(p-1)$$

Dividing both sides by p, $p \neq 0$, this simplifies to

$$\begin{aligned} 1.4(1.4p-1) &= 2(p-1) \\ 1.96p - 1.4 &= 2p - 2 \\ 0.6 &= 0.04p \\ 15 &= p \end{aligned}$$

Therefore, there were 15 people at the first meeting.

Problem of the Week

Problem E

Exponentially Large

Alex can choose four different numbers w, x, y and z from the set

$$\{-1, -2, -3, -4, -5\}$$

What is the largest possible value of $w^x + y^z$?

$$w^x + y^z$$

$w^x + y^z$

Problem of the Week

Problem E and Solution

Exponentially Large

Problem

Alex can choose four different numbers w, x, y and z from the set $\{-1, -2, -3, -4, -5\}$. What is the largest possible value of $w^x + y^z$?

Solution

Consider w^x and choose w and x to be different numbers from the set $\{-1, -2, -3, -4, -5\}$. What is the largest possible value for w^x?

Since x will be negative, we write $w^x = \dfrac{1}{w^{-x}}$, where $-x > 0$.

If x is odd, then since w is negative, then w^x will be negative.
If x is even, then w^x will be positive.
So to make w^x as large as possible, we make x even (ie. -2 or -4).

Also, in order to make $w^x = \dfrac{1}{w^{-x}}$ as large as possible, we want to make the denominator, w^{-x}, as small as possible, so w should be as small as possible in absolute value.

Therefore, the largest possible value of w^x will be when $w = -1$ and x is either -2 or -4, giving 1 in both cases (ie. $(-1)^{-2} = (-1)^{-4} = 1$).

What is the second largest possible value for w^x?
Again, we need x to be even to make w^x positive, and from above, we can assume that $w \neq -1$.
When $x = -2$, the smallest possible base (in absolute value) is $w = -3$ and $w^x = \dfrac{1}{(-3)^2} = \dfrac{1}{9}$.
When $x = -4$, the smallest possible base (in absolute value) is $w = -2$ and $w^x = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$.

The largest of these two is $\dfrac{1}{9}$.

Therefore, the two largest possible values for w^x are 1 and $\dfrac{1}{9}$.

Thus, looking at $w^x + y^z$, since -1 can only be chosen for one of these four numbers, then the largest possible value for this expression is the sum of the largest two possible values for w^x, ie. $1 + \dfrac{1}{9} = \dfrac{10}{9}$, which is obtained by calculating $(-1)^{-4} + (-3)^{-2}$.

Therefore, the largest value of $w^x + y^z$ is $\dfrac{10}{9}$. (This will occur when $w = -1$, $x = -4$, $y = -3$, and $z = -2$ or $w = -3$, $x = -2$, $y = -1$, and $z = -4$.)

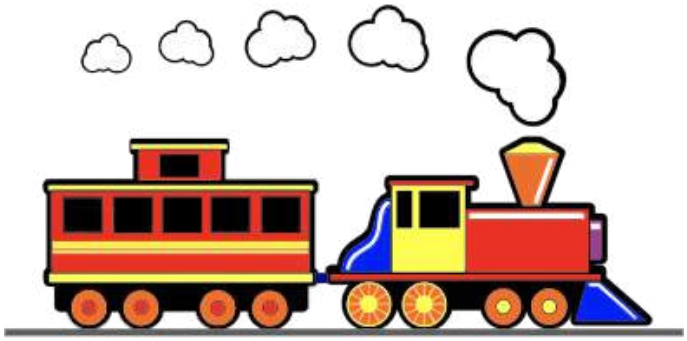
Problem of the Week

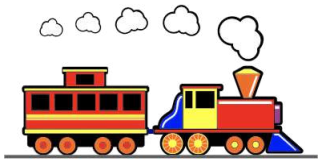
Problem E

Stay on Track

At noon, two trains are 1000 km apart. The first train is north of the second train and is traveling south. The second train is traveling east. Both trains travel at the same average speed, 100 km/h.

At what time is the **total** distance travelled by the two trains equal to the distance between the trains?





Problem of the Week

Problem E and Solution

Stay on Track

Problem

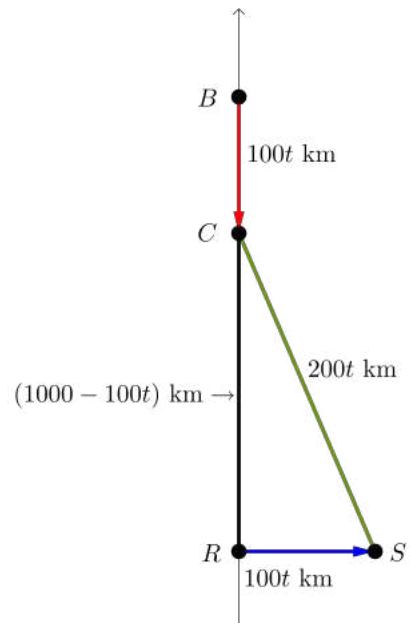
At noon, two trains are 1000 km apart. The first train is north of the second train and is traveling south. The second train is traveling east. Both trains travel at the same average speed, 100 km/h.
At what time is the **total** distance travelled by the two trains equal to the distance between the trains?

Solution

Let t be the time in hours until the total distance travelled by the two trains is equal to the distance between the trains.

Since each train is travelling at 100 km/h, the distance travelled by each train in t hours is $100t$ km. The total distance travelled by the two trains is $2 \times 100t = 200t$ km.

The diagram shows the positions of the two trains after t hours. The southbound train starts at B and moves to C. The eastbound train starts at R and moves to S. Then $BC = RS = 100t$ and since $BR = 1000$, $CR = 1000 - 100t$. We want the time t when $CS = BC + RS = 100t + 100t = 200t$.



$$\begin{aligned}
\triangle CRS \text{ is right angled, so} \qquad CS^2 &= CR^2 + RS^2 \\
(200t)^2 &= (1000 - 100t)^2 + (100t)^2 \\
40000t^2 &= 1000000 - 200000t + 10000t^2 + 10000t^2 \\
20000t^2 + 200000t - 1000000 &= 0 \\
t^2 + 10t - 50 &= 0 \\
\text{Using the Quadratic Formula,} \qquad t &= \frac{-10 \pm \sqrt{100 - 4(-50)}}{2} \\
t &= \frac{-10 \pm 10\sqrt{3}}{2} \\
t &= -5 \pm 5\sqrt{3}
\end{aligned}$$

Since $t > 0$, $-5 - 5\sqrt{3}$ is inadmissible. Therefore, $t = -5 + 5\sqrt{3} \doteq 3.66$ hours.

The distance between the two trains will be equal to their total distance travelled in $(-5 + 5\sqrt{3})$ hours, which is approximately 3 hours and 40 minutes after they leave their initial positions. The time will be approximately 3:40 pm.

Could the diagram be drawn any other way? On the next page the other two possible diagrams are briefly discussed.

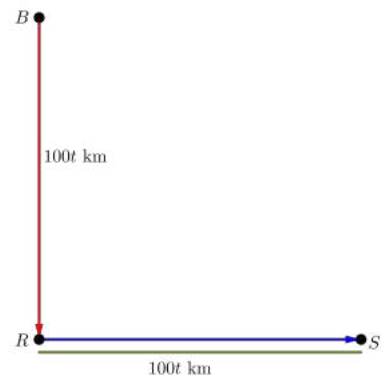
We will repeat some of the initial work and then apply it to a second diagram.

Let t be the time in hours until the total distance travelled by the two trains is equal to the distance between the trains.

Since each train is travelling at 100 km/h, the distance travelled by each train in t hours is $100t$ km. The total distance travelled by the two trains is $2 \times 100t = 200t$ km.

Maybe the train travelling south gets to a point in line with the west to east direction of the second train.

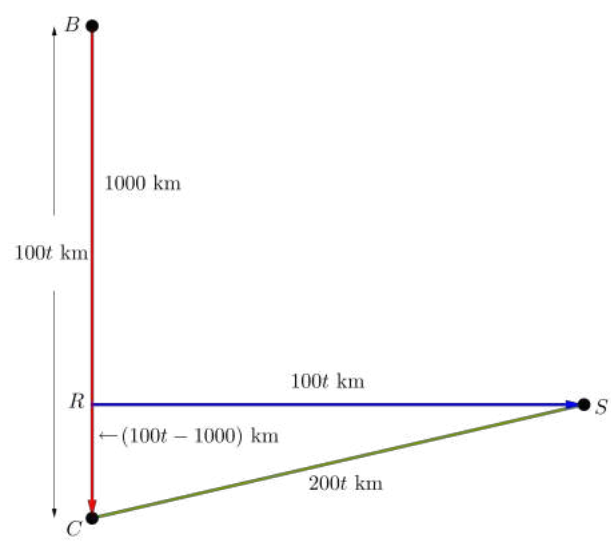
The diagram shows the positions of the two trains after t hours. Here, $BR = 1000$ km and it follows that $t = 10$ hours. The second train travels from R to S, a total of 1000 km, in the same time. But here RS is also the distance between the two trains, so is also supposed to be equal to the total distance travelled by the two trains. The total distance travelled is 2000 km, but $RS = 1000$ km. This diagram is not possible.



Is it possible that the train travelling south goes lower than the west to east line along which the second train travels?

The diagram shows the positions of the two trains after t hours. The distance travelled by the first train is represented by $BC = 100t$ km and it follows that $RC = (100t - 1000)$ km. The second train travels from R to S, a total of $100t$ km in the same time. The total distance is $CS = 200t$ km.

In any triangle, the sum of the lengths of any two sides of the triangle must be greater than the length of the third side. This is known as the triangle inequality.



In $\triangle RSC$, $RC + RS = (100t - 1000) + 100t = 200t - 1000 < 200t = CS$.

Here, the triangle inequality is not satisfied, so this "triangle" cannot represent the situation presented in this problem. It can be shown that the triangle shown on the previous page satisfies the triangle inequality.

Problem of the Week

Problem E

Can Drive On!

Bryn observed a real need in her community for canned food at the local food bank. On April 15, she decided to start a canned food drive, and hoped to collect 4000 cans of food by the end of June. She posted flyers and spread the word. She made the following observations.

Day Number	Total Number of Cans Collected Since Beginning of Drive	Increase from Previous Day
1	2	
2	5	3
3	12	7
4	23	11
5	38	15

Bryn noticed that the increases from one day to the next form an arithmetic sequence with first term 3 and common difference 4. (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ is an arithmetic sequence with four terms and common difference 2.)

Assuming that the pattern of daily increases continues, how many days would it take to collect at least 4000 cans of food?

The information on the next page *may* be helpful in solving the problem.

The sequence $3, 5, 7, 9$ is an arithmetic sequence with four terms and common difference 2. The term in position n is denoted t_n. For example, we say that $t_1 = 3$. The subscript 1 is the position of the term in the sequence and 3 is the value of the term.

The general term of an arithmetic sequence is $t_n = a + (n-1)d$, where a is the first term, d is the common difference, and n is the term number.

The sum, S_n, of the first n terms of an arithmetic sequence can be found using either $S_n = \frac{n}{2}(2a + (n-1)d)$ or $S_n = n\left(\frac{t_1 + t_n}{2}\right)$, where t_1 is the first term of the sequence and t_n is the nth term of the sequence.

For example, for the arithmetic sequence $3, 5, 7, 9$, we have $a = t_1 = 3$, $d = 2$, $t_4 = 9$, and $S_4 = 3 + 5 + 7 + 9 = 24$.

Also,

$$\begin{aligned}
\frac{4}{2}(2a + (4-1)d) &= \frac{4}{2}(2(3) + (4-1)2) \\
&= 2(12) \\
&= 24 \\
&= S_4
\end{aligned}$$

And,

$$\begin{aligned}
4\left(\frac{t_1 + t_4}{2}\right) &= 4\left(\frac{3+9}{2}\right) \\
&= 4(6) \\
&= 24 \\
&= S_4
\end{aligned}$$

Problem of the Week
Problem E and Solution
Can Drive On!

Problem

Bryn observed a real need in her community for canned food at the local food bank. On April 15, she decided to start a canned food drive, and hoped to collect 4000 cans of food by the end of June. She posted flyers and spread the word. She made the following observations.

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Bryn noticed that the increases from one day to the next form an arithmetic sequence with first term 3 and common difference 4. (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ is an arithmetic sequence with four terms and common difference 2.) Assuming that the pattern of daily increases continues, how many days would it take to collect at least 4000 cans of food?

Solution

Solution 1

On day i, let the total number of cans collected since the beginning of the drive be a_i. That is, $a_1 = 2$, $a_2 = 5$, $a_3 = 12$, $a_4 = 23$, and $a_5 = 38$. We want to determine the day number, n, so that $a_n \geq 4000$.

Let the arithmetic sequence of the increases from day to day be $t_1,\ t_2,\ t_3,\ \cdots,\ t_n,\ \cdots$.

Then $t_1 = a_2 - a_1 = 5 - 2 = 3$, $t_2 = a_3 - a_2 = 12 - 5 = 7$, $t_3 = a_4 - a_3 = 23 - 12 = 11$, and $t_4 = a_5 - a_4 = 38 - 23 = 15$. This sequence of increases is arithmetic, with common difference $d = t_2 - t_1 = 7 - 3 = 4$. To determine t_5, we add the common difference 4 to t_4. So $t_5 = t_4 + 4 = 15 + 4 = 19$. Then $a_6 = a_5 + t_5 = 38 + 19 = 57$. This means that the total number of cans on day 6 is 57. We could continue generating these increases and number of cans on the next day until the goal is reached.

Let's take a closer look at the number of cans.

$a_1 = 2$
$a_2 = a_1 + t_1 = 2 + 3 = 5$
$a_3 = a_2 + t_2 = 2 + 3 + 7 = 12$
$a_4 = a_3 + t_3 = 2 + 3 + 7 + 11 = 23$
$a_5 = a_4 + t_4 = 2 + 3 + 7 + 11 + 15 = 38$
$a_6 = a_5 + t_5 = 2 + 3 + 7 + 11 + 15 + 19 = 57$

We can present the information in a table.

Day Number n	Total Number of Cans a_n	Daily Increase t_n	Difference of the Daily Increase d
1	2	3	
2	5	7	4
3	12	11	4
4	23	15	4
5	38	19	4
6	57		

Since the second difference is constant, we can represent the general term of the sequence of total cans with a quadratic function in n. Let $a_n = pn^2 + qn + r$, where p, q, r are constants.

For $n = 1$, $a_1 = 2 = p(1)^2 + q(1) + r$. Therefore, $p + q + r = 2$. (1)
For $n = 2$, $a_2 = 5 = p(2)^2 + q(2) + r$. Therefore, $4p + 2q + r = 5$. (2)
For $n = 3$, $a_3 = 12 = p(3)^2 + q(3) + r$. Therefore, $9p + 3q + r = 12$. (3)

Subtracting (1) from (2), $3p + q = 3$. (4)
Subtracting (2) from (3), $5p + q = 7$. (5)

Subtracting (4) from (5), $2p = 4$ and $p = 2$ follows.
Substituting $p = 2$ into (4), $3(2) + q = 3$ and $q = -3$ follows.

Substituting $p = 2,\ q = -3$ into (1), $2 - 3 + r = 2$ and $r = 3$ follows.

Therefore, $a_n = 2n^2 - 3n + 3$ is the general term of the sequence of total cans, in terms of n.

We want to find the value of n so that $2n^2 - 3n + 3 \geq 4000$. We will do so by first solving $2n^2 - 3n + 3 = 4000$ for n. Rearranging the equation, we obtain $2n^2 - 3n - 3997 = 0$. Using the quadratic formula, $n \approx -43.96$ or $n \approx 45.46$. Since n is the day number, $n > 0$. Therefore, $n \approx 45.46$, and it follows that on day 46 there would be over 4000 cans collected in total.

If we check when $n = 45$, $a_{45} = 3918$ cans, which is under the goal.

When $n = 46$, $a_{46} = 4097$ cans and the goal is achieved. Therefore, it would take 46 days to collect at least 4000 cans of food. There are over 46 days from April 15 to the end of June, so it is possible to achieve the goal if the pattern continues.

Two more solutions follow.

Solution 2

As in Solution 1, on day i, let the total number of cans collected since the beginning of the drive be a_i and let the sequence of daily increases be $t_1,\ t_2,\ t_3,\ \cdots,\ t_n,\ \cdots$. Then $t_1 = a_2 - a_1 = 3$ and $t_2 = a_3 - a_2 = 7$. Since the sequence of daily increases is arithmetic, the constant difference is $d = 7 - 3 = 4$. We can generate more terms: $t_3 = 11,\ t_4 = 15,\ t_5 = 19,\ \cdots$.

Each term in the sequence of daily increases is the difference between consecutive terms of the original sequence, so we get the following equations.

$$\begin{aligned} a_2 - a_1 &= t_1 \\ a_3 - a_2 &= t_2 \\ a_4 - a_3 &= t_3 \\ &\vdots \\ a_{n-2} - a_{n-3} &= t_{n-3} \\ a_{n-1} - a_{n-2} &= t_{n-2} \\ a_n - a_{n-1} &= t_{n-1} \end{aligned}$$

Adding these equations, we get

$$a_n - a_1 = t_1 + t_2 + t_3 + \cdots + t_{n-2} + t_{n-1} = S_{n-1} \quad (1)$$

as illustrated below.

$$\begin{array}{rrcrcl} & \cancel{a_2} & - & a_1 & = & t_1 \\ & \cancel{a_3} & - & \cancel{a_2} & = & t_2 \\ & \cancel{a_4} & - & \cancel{a_3} & = & t_3 \\ & \vdots & & \vdots & & \vdots \\ & \cancel{a_{n-2}} & - & \cancel{a_{n-3}} & = & t_{n-3} \\ & \cancel{a_{n-1}} & - & \cancel{a_{n-2}} & = & t_{n-2} \\ + & a_n & - & \cancel{a_{n-1}} & = & t_{n-1} \\ \hline & a_n & - & a_1 & = & t_1 + t_2 + t_3 + \cdots + t_{n-2} + t_{n-1} \quad (1) \end{array}$$

To find the sum $S_{n-1} = t_1 + t_2 + t_3 + \cdots + t_{n-2} + t_{n-1}$, we can use the formula $S_n = \frac{n}{2}[2a + (n-1)d]$ with $a = 3$ and $d = 4$ for $(n-1)$ terms. So,

$$\begin{aligned} S_{n-1} &= t_1 + t_2 + t_3 + \cdots + t_{n-2} + t_{n-1} \\ &= \frac{n-1}{2}[2a + ((n-1)-1)d] \\ &= \frac{n-1}{2}[2(3) + ((n-1)-1)(4)] \\ &= \frac{n-1}{2}[6 + (n-2)(4)] \\ &= \frac{n-1}{2}[6 + 4n - 8] \\ &= \frac{n-1}{2}[4n - 2] \\ &= (n-1)(2n-1) \end{aligned}$$

From (1), $a_n - a_1 = S_{n-1}$, so $a_n = S_{n-1} + a_1 = (n-1)(2n-1) + 2 = 2n^2 - 3n + 3$.

Therefore, $a_n = 2n^2 - 3n + 3$ is the general term of the sequence of total cans, in terms of n.

Then, as in Solution 1, we would find the value of n so that $2n^2 - 3n + 3 \geq 4000$. Without repeating the work here, we find that it would take 46 days to collect at least 4000 cans of food.

Solution 3

As in Solution 1, on day i, let the total number of cans collected since the beginning of the drive be a_i and let the sequence of daily increases be $t_1,\ t_2,\ t_3,\ \cdots,\ t_n,\ \cdots$. Then $t_1 = a_2 - a_1 = 3$ and $t_2 = a_3 - a_2 = 7$. Since the sequence of daily increases is arithmetic, the constant difference is $d = 7 - 3 = 4$. Using the formula for the general term of an arithmetic sequence, $t_n = a + (n-1)d$, the general term of the sequence of increases is $t_n = 3 + (n-1)(4) = 3 + 4n - 4 = 4n - 1$.

To generate the sequence of total cans, we start with the first term and add more terms from the arithmetic sequence of daily increases. For example,
$a_1 = 2$
$a_2 = 2 + t_1 = 2 + [4(1) - 1] = 2 + 3 = 5$
$a_3 = 2 + t_1 + t_2 = 2 + [4(1) - 1] + [4(2) - 1] = 2 + 3 + 7 = 12$

So,

$$\begin{aligned}
a_n &= 2 + t_1 + t_2 + t_3 + \cdots + t_{n-1} \\
&= 2 + [4(1) - 1] + [4(2) - 1] + [4(3) - 1] + \cdots + [4(n-1) - 1] \\
&= 2 + [4(1) + 4(2) + 4(3) + \cdots + 4(n-1)] + (n-1)(-1) \\
&= 2 + 4[1 + 2 + 3 + \cdots + (n-1)] - n + 1 \\
&= 3 - n + 4\left[\frac{(n-1)(n)}{2}\right] \qquad (1) \\
&= 3 - n + 2(n^2 - n) \\
&= 3 - n + 2n^2 - 2n \\
&= 2n^2 - 3n + 3
\end{aligned}$$

An explanation is provided here to show how (1) is obtained.
Notice that $1 + 2 + 3 + \cdots + (n-1)$ is an arithmetic sequence with $(n-1)$ terms, first term $b_1 = 1$, and last term $b_{n-1} = (n-1)$. Using the formula for the sum of the terms of an arithmetic sequence, $S_n = n\left(\frac{b_1 + b_n}{2}\right)$, we obtain

$$S_{n-1} = (n-1)\left(\frac{1 + (n-1)}{2}\right) = (n-1)\left(\frac{n}{2}\right) = \frac{(n-1)(n)}{2}$$

Therefore, $a_n = 2n^2 - 3n + 3$ is the general term of the sequence of total cans, in terms of n.

Then, as in Solution 1, we would find the value of n so that $2n^2 - 3n + 3 \geq 4000$. Without repeating the work here, we find that it would take 46 days to collect at least 4000 cans of food.

Problem of the Week

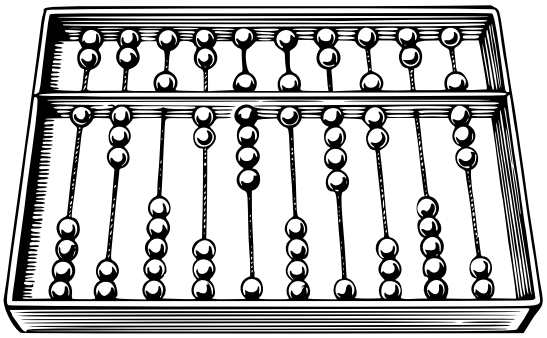
Problem E

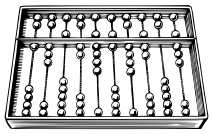
Count on That

Let n be a positive integer. How many values of n satisfy the following inequality?

$$(n-1)(n-3)(n-5)\cdots(n-2019)(n-2021) \leq 0$$

NOTE: The product on the left side of the inequality consists of 1011 factors of the form $n-d$, where the value of d starts at 1 and increases by 2 for each subsequent factor.





Problem of the Week
Problem E and Solution
Count on That

Problem
Let n be a positive integer. How many values of n satisfy the following inequality?

$$(n-1)(n-3)(n-5)\cdots(n-2019)(n-2021) \leq 0$$

NOTE: The product on the left side of the inequality consists of 1011 factors of the form $n-d$, where the value of d starts at 1 and increases by 2 for each subsequent factor.

Solution
We will consider two cases. First when the product on the left side *equals* zero, and then when the product on the left side *is less than* zero.

Case 1: $(n-1)(n-3)(n-5)\cdots(n-2019)(n-2021) = 0$
The product of factors on the left side equals zero when any one of the factors is equal to zero. This happens when $n = 1,\ 3,\ 5, \ldots,\ 2019$, or 2021. These are all the odd integers between 1 and 2021, inclusive. The number of these integers is equal to $\frac{2021+1}{2} = 1011$.

Case 2: $(n-1)(n-3)(n-5)\cdots(n-2019)(n-2021) < 0$
The product of factors on the left side is less than zero (i.e. negative) when none of the factors are equal to zero and an odd number of the factors are negative. Note that for every integer n, the following is true.

$$n-1 > n-3 > n-5 > \cdots > n-2019 > n-2021$$

Now we notice the following.

- When $n = 2$, it follows that $n-1 = 1,\ \ n-3 = -1$, and so the remaining factors will all be negative. This is a total of $1011 - 1 = 1010$ negative factors. Since this is an even number, the product of factors will be positive.
- When $n = 4$, it follows that $n-1 = 3,\ \ n-3 = 1,\ \ n-5 = -1$, and so the remaining factors will all be negative. This is a total of $1011 - 2 = 1009$ negative factors. Since this is an odd number, the product of factors will be negative.
- When $n = 6$, it follows that $n-1 = 5,\ \ n-3 = 3,\ \ n-5 = 1,\ \ n-7 = -1$, and so the remaining factors will all be negative. This is a total of $1011 - 3 = 1008$ negative factors. Since this is an even number, the product of factors will be positive.
- When $n = 8$, it follows that $n-1 = 7,\ \ n-3 = 5,\ \ n-5 = 3,\ \ n-7 = 1,\ \ n-9 = -1$, and so the remaining factors will all be negative. This is a total of $1011 - 4 = 1007$ negative factors. Since this is an odd number, the product of factors will be negative.

This pattern continues, giving a negative product of factors when
$n = 4,\ 8,\ 12,\ 16, \ldots, 2016,\ 2020$. Notice that these are all the multiples of 4 that are less than 2021. Since $2020 \div 4 = 505$, that tells us there are 505 such values.

When $n > 2021$, all the factors will be positive, and thus their product will also be positive.

Therefore, in total there are $1011 + 505 = 1516$ values of n that satisfy the inequality.

Problem of the Week

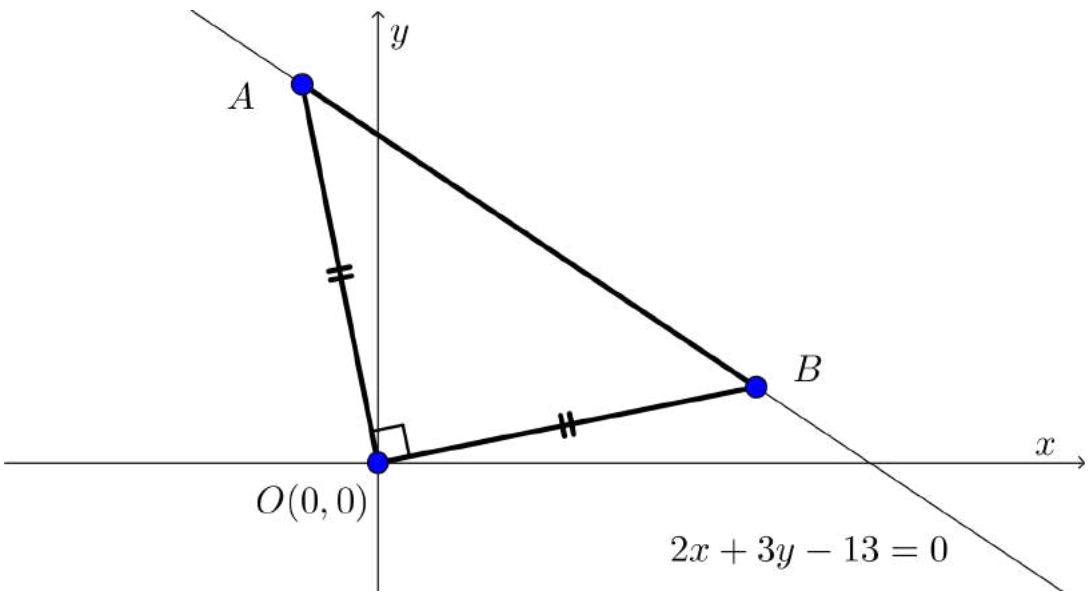
Problem E

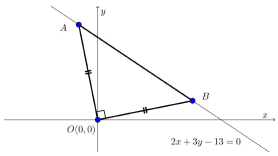
The Hypotenuse is Aligned

$\triangle OAB$ is an isosceles right-angled triangle with

- vertex O located at the origin; and
- vertices A and B located on the line $2x + 3y - 13 = 0$ such that $\angle AOB = 90^\circ$ and $OA = OB$.

Determine the area of $\triangle OAB$.





Problem of the Week

Problem E and Solution

The Hypotenuse is Aligned

Problem

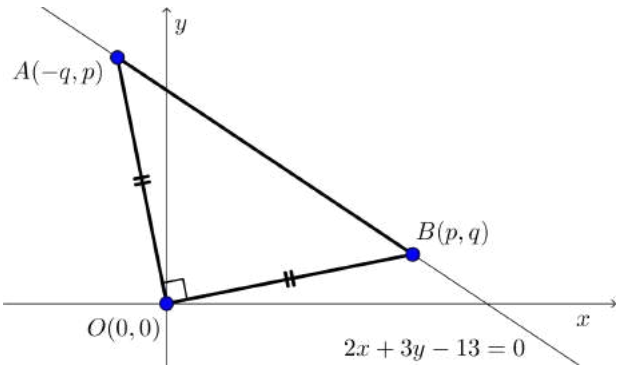
$\triangle OAB$ is an isosceles right-angled triangle with

- vertex O located at the origin; and
- vertices A and B located on the line $2x + 3y - 13 = 0$ such that $\angle AOB = 90^\circ$ and $OA = OB$.

Determine the area of $\triangle OAB$.

Solution

Solution 1



Let B have coordinates (p, q). Then the slope of $OB = \frac{q}{p}$. Since $\angle AOB = 90^\circ$, then $OB \perp OA$ and the slope of OA is the negative reciprocal of the slope of OB. Therefore, the slope of $OA = \frac{p}{-q}$. Since the triangle is isosceles with $OA = OB$, it follows that the coordinates of A are $(-q, p)$. (We can verify this by finding the length of OA and the length of OB and showing that both lengths are equal to $\sqrt{p^2 + q^2}$.)

Since $B(p, q)$ lies on the line $2x + 3y - 13 = 0$, it satisfies the equation of the line.
Therefore, $2p + 3q - 13 = 0 \quad (1)$.

Since $A(-q, p)$ lies on the line $2x + 3y - 13 = 0$, it satisfies the equation of the line.
Therefore, $-2q + 3p - 13 = 0$, or $3p - 2q - 13 = 0 \quad (2)$.

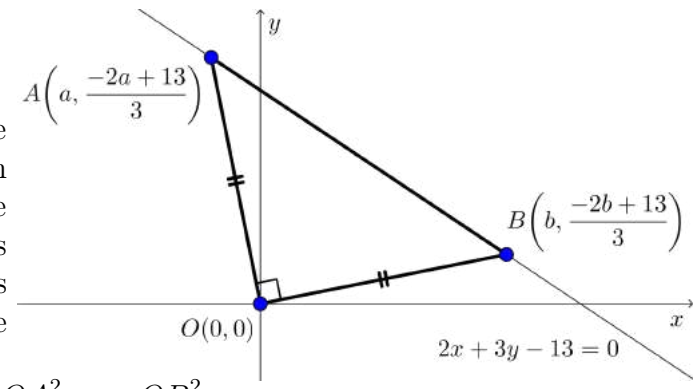
Since we have two equations and two unknowns, we can use elimination to solve for p and q.

$$\begin{aligned}
(1) \times 2: \quad 4p + 6q - 26 &= 0 \\
(2) \times 3: \quad 9p - 6q - 39 &= 0 \\
\text{Adding, we obtain}: \quad 13p - 65 &= 0 \\
p &= 5 \\
\text{Substituting in (1)}: \quad 10 + 3q - 13 &= 0 \\
3q &= 3 \\
q &= 1
\end{aligned}$$

Therefore, the point B is $(5, 1)$ and the length of OB is $\sqrt{5^2 + 1^2} = \sqrt{26}$. Since $OA = OB$, $OA = \sqrt{26}$.
$\triangle AOB$ is a right-angled triangle, so we can use OB as the base and OA as the height in the formula for the area of a triangle. Therefore, the area of $\triangle AOB$ is $\dfrac{OA \times OB}{2} = \dfrac{\sqrt{26}\sqrt{26}}{2} = 13$.

Therefore, the area of $\triangle AOB$ is 13 units^2.

Solution 2



By rearranging the given equation for the line, we obtain $y = \frac{-2x+13}{3}$. Since the points A and B are on the line, their coordinates satisfy the equation of the line. If A has x-coordinate a, then A has coordinates $\left(a, \frac{-2a+13}{3}\right)$. If B has x-coordinate b, then B has coordinates $\left(b, \frac{-2b+13}{3}\right)$. Since $\triangle OAB$ is isosceles, we know that $OA = OB$. Then

$$\begin{aligned}
OA^2 &= OB^2 \\
a^2 + \left(\frac{-2a+13}{3}\right)^2 &= b^2 + \left(\frac{-2b+13}{3}\right)^2 \\
a^2 + \frac{4a^2 - 52a + 169}{9} &= b^2 + \frac{4b^2 - 52b + 169}{9} \\
\text{Multiplying by 9 :} \quad 9a^2 + 4a^2 - 52a + 169 &= 9b^2 + 4b^2 - 52b + 169 \\
\text{Simplifying :} \quad 13a^2 - 52a + 169 &= 13b^2 - 52b + 169 \\
\text{Rearranging :} \quad 13a^2 - 13b^2 - 52a + 52b &= 0 \\
\text{Dividing by 13 :} \quad a^2 - b^2 - 4a + 4b &= 0 \\
\text{Factoring pairs :} \quad (a+b)(a-b) - 4(a-b) &= 0 \\
\text{Common factoring :} \quad (a-b)(a+b-4) &= 0
\end{aligned}$$

Solving, $a = b$ or $a = 4 - b$. Since A and B are distinct points, $a \neq b$. Therefore, $a = 4 - b$. We can rewrite $A\left(a, \frac{-2a+13}{3}\right)$ as $A\left(4-b, \frac{-2(4-b)+13}{3}\right)$ which simplifies to $A\left(4-b, \frac{2b+5}{3}\right)$.

Since $\triangle OAB$ is a right-angled triangle, we can use the Pythagorean Theorem, and $AB^2 = OA^2 + OB^2$ follows. But $OA = OB$, so this can be written $AB^2 = 2OB^2$.

$$\begin{aligned}
AB^2 &= 2OB^2 \\
(b-(4-b))^2 + \left(\frac{-2b+13}{3} - \frac{2b+5}{3}\right)^2 &= 2\left[b^2 + \left(\frac{-2b+13}{3}\right)^2\right] \\
(2b-4)^2 + \left(\frac{-4b+8}{3}\right)^2 &= 2\left[b^2 + \frac{4b^2 - 52b + 169}{9}\right] \\
4b^2 - 16b + 16 + \frac{16b^2 - 64b + 64}{9} &= 2b^2 + \frac{8b^2 - 104b + 338}{9} \\
\text{Multiplying by 9 :} \quad 36b^2 - 144b + 144 + 16b^2 - 64b + 64 &= 18b^2 + 8b^2 - 104b + 338 \\
\text{Simplifying :} \quad 52b^2 - 208b + 208 &= 26b^2 - 104b + 338 \\
\text{Rearranging :} \quad 26b^2 - 104b - 130 &= 0 \\
\text{Dividing by 26 :} \quad b^2 - 4b - 5 &= 0 \\
\text{Factoring :} \quad (b-5)(b+1) &= 0
\end{aligned}$$

It follows that $b = 5$ or $b = -1$. When $b = 5$, the point A is $(-1, 5)$ and the point B is $(5, 1)$. When $b = -1$, the point A is $(5, 1)$ and the point B is $(-1, 5)$. There are only two points. The area calculations shown in Solution 1 follow from here.

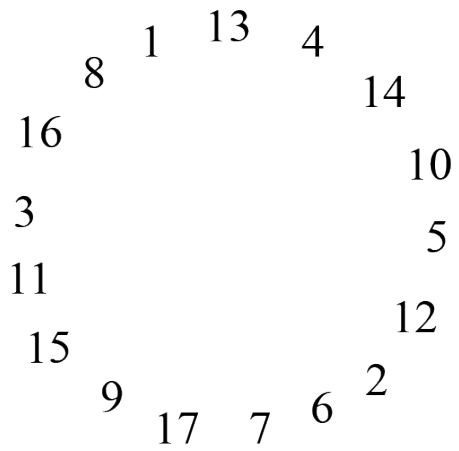
Therefore, the area of $\triangle OAB$ is 13 units^2.

Problem of the Week

Problem E

Stand in a Circle

The numbers from 1 to 17 are arranged around a circle. One such arrangement is shown.



Explain why every possible arrangement of these numbers around a circle must have at least one group of three adjacent numbers whose sum is at least 27.

NOTE:
In solving the above problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

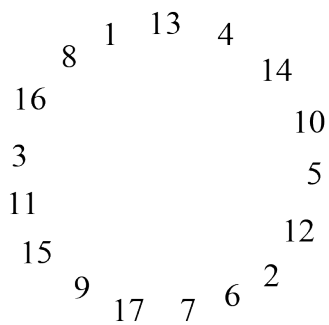
Problem of the Week

Problem E and Solution

Stand in a Circle

Problem

The numbers from 1 to 17 are arranged around a circle. One such arrangement is shown.



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In solving the above problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

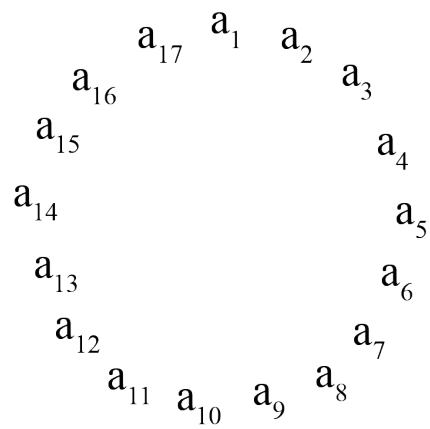
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Solution

We will use a *proof by contradiction* to explain why every possible arrangement of these numbers around a circle must have at least one group of three adjacent numbers whose sum is at least 27.

In general, to prove that a statement is true using a *proof by contradiction*, we first assume the statement is false. We then show this leads to a contradiction, which proves that our original assumption was wrong, and therefore the statement must be true.

First, we will assume that there exists an arrangement of the numbers from 1 to 17 around a circle where the sums of all groups of three adjacent numbers are less than 27. This arrangement is shown, where the variables $a_1,\ a_2,\ a_3, \ldots, a_{17}$ represent the numbers from 1 to 17, in some order, for this particular arrangement.



Now we will add up the sums of all groups of three adjacent numbers and call this value S.

$$\begin{aligned} S =& (a_1 + a_2 + a_3) + (a_2 + a_3 + a_4) + (a_3 + a_4 + a_5) \\ &+ (a_4 + a_5 + a_6) + (a_5 + a_6 + a_7) + (a_6 + a_7 + a_8) \\ &+ (a_7 + a_8 + a_9) + (a_8 + a_9 + a_{10}) + (a_9 + a_{10} + a_{11}) \\ &+ (a_{10} + a_{11} + a_{12}) + (a_{11} + a_{12} + a_{13}) + (a_{12} + a_{13} + a_{14}) \\ &+ (a_{13} + a_{14} + a_{15}) + (a_{14} + a_{15} + a_{16}) + (a_{15} + a_{16} + a_{17}) \\ &+ (a_{16} + a_{17} + a_1) + (a_{17} + a_1 + a_2) \end{aligned}$$

We can see that there are 17 groups of three adjacent numbers around the circle. Since each of these groups has a sum that is less than 27, we can conclude that S must be less than $17 \times 27 = 459$. So, $S < 459$.

Looking again at the value of S, we can see that each of $a_1,\ a_2,\ a_3, \ldots, a_{17}$ appears exactly three times. So,

$$\begin{aligned} S &= 3(a_1) + 3(a_2) + 3(a_3) + \cdots + 3(a_{17}) \\ &= 3(a_1 + a_2 + a_3 + \cdots + a_{17}) \end{aligned}$$

However, we know that $a_1 + a_2 + a_3 + \cdots + a_{17}$ is equal to the sum of all the numbers from 1 to 17, which is $\frac{17(18)}{2} = 153$. Therefore, $S = 3(153) = 459$.

But this is a contradiction, since we stated earlier that $S < 459$. It can't be possible that $S < 459$ and $S = 459$. Therefore, our original assumption that there exists an arrangement of the numbers from 1 to 17 around a circle where the sums of all groups of three adjacent numbers are less than 27 must be false. Thus, it follows that every possible arrangement of the numbers from 1 to 17 around a circle must have at least one group of three adjacent numbers whose sum is at least 27.

Data Management (D)

TAKE ME TO THE COVER

Problem of the Week

Problem E

Such a Card

Johanna has a deck of cards with the following properties:

1. Each card in the deck has a positive three-digit integer on it.
2. There is exactly one card in the deck for every three-digit positive integer.

Johanna randomly selects a card from the deck of cards. Determine the probability that the sum of the digits of the integer on this card is 15.

456

Problem of the Week
Problem E and Solution
Such a Card

Problem

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1. Each card in the deck has a positive three-digit integer on it.
2. There is exactly one card in the deck for every three-digit positive integer.

Johanna randomly selects a card from the deck of cards. Determine the probability that the sum of the digits of the integer on this card is 15.

Solution

To begin, we need to determine the number of cards in the deck. Since there is a card for each three-digit positive integer, there are 900 cards in the deck. We must be careful calculating this number. There are 999 positive integers less than 1000. Of this set, 90 are two-digit numbers and 9 are single-digit numbers. Therefore there are $999 - 90 - 9 = 900$ three-digit positive integers.

Next we need to determine the digit combinations on a card that have a sum of 15. We will determine the possibilities using cases. Then we will look at the specific groups of numbers that sum to 15 to count the number of cards produced from each group.

1. One of the digits on the card is a 0. The other two digits on the card must add to 15. This leads to two groups of numbers: $(0, 6, 9)$ and $(0, 7, 8)$.
2. One of the digits on the card is a 1 but the number does not contain a 0. The other two digits on the card must add to 14. This leads to three groups of numbers: $(1, 5, 9)$, $(1, 6, 8)$ and $(1, 7, 7)$.
3. One of the digits on the card is a 2 but the number does not contain a 0 or 1. The other two digits on the card must add to 13. This leads to three groups of numbers: $(2, 4, 9)$, $(2, 5, 8)$ and $(2, 6, 7)$.
4. One of the digits on the card is a 3 but the number does not contain a 0, 1, or 2. The other two digits on the card must add to 12. This leads to four groups of numbers: $(3, 3, 9)$, $(3, 4, 8)$, $(3, 5, 7)$ and $(3, 6, 6)$.
5. One of the digits on the card is a 4 but the number does not contain a 0, 1, 2, or 3. The other two digits on the card must add to 11. This leads to two groups of numbers: $(4, 4, 7)$ and $(4, 5, 6)$.
6. One of the digits on the card is a 5 but the number does not contain a 0, 1, 2, 3, or 4. The other two digits on the card must add to 10. This leads to only one group of numbers: $(5, 5, 5)$.

Now that we have the groups of numbers, we can determine the number of cards that can be created from each group of three numbers. We will do this again with cases: groups containing a 0, groups containing three distinct numbers but not 0, groups containing exactly two numbers the same but not 0, and groups containing three numbers the same but not 0.

1. One of the numbers on the card is 0. Earlier we found that there were two such groups: $(0, 6, 9)$ and $(0, 7, 8)$. This is a special case since 0 cannot appear in the number as the hundreds digit for the number to be a three-digit number. For each of the two groups of numbers, the 0 can be placed in two ways, in the tens digit or the units digit. Once the 0 is placed, the other two numbers can be placed in the remaining two spots in two ways. Each group can form $2 \times 2 = 4$ three-digit numbers. Since there are two groups, there are $2 \times 4 = 8$ cards in the deck that contain a 0 and add to 15.

2. All three digits on the card are different and the number does not contain a 0. From the earlier cases there are eight groups in which all three numbers are different: $(1, 5, 9)$, $(1, 6, 8)$, $(2, 4, 9)$, $(2, 5, 8)$, $(2, 6, 7)$, $(3, 4, 8)$, $(3, 5, 7)$, and $(4, 5, 6)$. For each of these groups, the hundreds digit can be filled three ways. For each of these three choices for hundreds digit, the tens digit can be filled two ways. Once the hundreds digit and tens digit are selected, the units digit must get the third number. So each group can form $3 \times 2 = 6$ different numbers. Since there are eight groups, there are $8 \times 6 = 48$ cards in the deck that contain three different digits other than 0 that add to 15.

3. Two of the digits on the card are the same and the number does not contain a 0. From the earlier cases, there are four groups of numbers in which exactly two of the numbers in the group are the same: $(1, 7, 7)$, $(3, 3, 9)$, $(3, 6, 6)$, and $(4, 4, 7)$. For each of these groups, the unique number can be placed in one of three spots. Once the unique number is placed the other two numbers must go in the remaining two spots. So each group can form three different numbers. Since there are four groups, there are $4 \times 3 = 12$ cards in the deck that do not contain a 0 but contain two digits the same and whose digits add to 15.

4. The three digits on the card are the same. From the earlier cases we discovered only one such group: $(5, 5, 5)$. Only one card can be produced using the numbers from this group.

Combining the counts from the above four cases, there are $8 + 48 + 12 + 1 = 69$ cards in the deck with a digit sum of 15. Therefore, the probability that Johanna selects card whose digits add to 15 is $\frac{69}{900} = \frac{23}{300}$. This translates to approximately a 7.7% chance.

A game is considered *fair* if there is close to a 50% chance of winning. If Johanna was playing a game where she can win by drawing a card whose digits sum to 15, then this game is definitely not fair. If you changed the game to "if the card chosen has a sum that is divisible by 5", there is about a 20% chance of winning. This is better but still not fair.

Can you create a game using this specific deck of cards that is reasonably fair and fun to play?

Problem of the Week

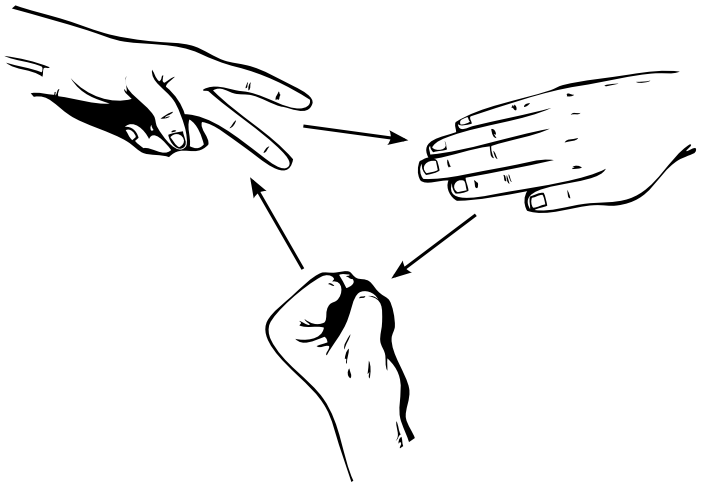
Problem E

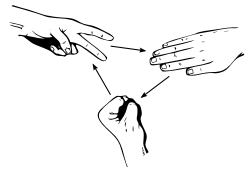
Rock Out

Sameer likes playing rock paper scissors so much that he started a club at his school. At every club meeting, each person plays rock paper scissors against each of the other people at the meeting the same number of times.

There were 40% more people at the second club meeting than the first. The total number of rock paper scissors matches played at the second meeting was twice the number of matches played at the first meeting. The number of matches each person plays against each of the other people at the meeting stays the same from meeting to meeting.

How many people were at the first meeting?





Problem of the Week
Problem E and Solution
Rock Out

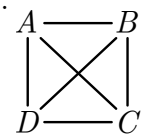
Problem

Sameer likes playing rock paper scissors so much that he started a club at his school. At every club meeting, each person plays rock paper scissors against each of the other people at the meeting the same number of times. There were 40% more people at the second club meeting than the first. The total number of rock paper scissors matches played at the second meeting was twice the number of matches played at the first meeting. The number of matches each person plays against each of the other people at the meeting stays the same from meeting to meeting. How many people were at the first meeting?

Solution

Let p represent the number of people at the first meeting. Then the number of people at the second meeting is $1.4p$. Note that both p and $1.4p$ must be positive integers.

We need to first establish how many games are played. Suppose there were 4 people ($A, B, C,$ and D), and each person played against every other person once, as shown to the right.



We can see there would be 6 matches played ($AB, AC, AD, BC, BD,$ and CD). This is easy to see in a small diagram, but as the number of people and matches increase, the diagrams become hard to read, so we need to find a general solution.

Often, when counting something like this we "double-count" by mistake. We think that because there are 4 people and each person plays the 3 other people, that means there are $4 \times 3 = 12$ matches in total. However, we have counted each match twice, so we need to divide the result by 2. So if there were 4 people, with each person playing every other person once, there would be $\frac{4\times 3}{2} = 6$ matches played in total.

In general, if there are p people and each person plays against every other person once, there would be $\frac{p(p-1)}{2}$ matches played in total. If each person plays against every other person n times, the total number of matches would need to be multiplied by n, to obtain $n\left(\frac{p(p-1)}{2}\right)$. This represents the total number of matches at the first meeting. Now in the second meeting, the number of people is $1.4p$, so the total number of matches in the second meeting would be $n\left(\frac{1.4p(1.4p-1)}{2}\right)$.

We know that the number of matches played at the second meeting is twice the number of matches played at the first meeting. So,

$$n\left(\frac{1.4p(1.4p-1)}{2}\right) = 2\left[n\left(\frac{p(p-1)}{2}\right)\right]$$

Dividing both sides by $\frac{n}{2}$, $n \neq 0$, this simplifies to

$$1.4p(1.4p-1) = 2p(p-1)$$

Dividing both sides by p, $p \neq 0$, this simplifies to

$$\begin{aligned} 1.4(1.4p-1) &= 2(p-1) \\ 1.96p - 1.4 &= 2p - 2 \\ 0.6 &= 0.04p \\ 15 &= p \end{aligned}$$

Therefore, there were 15 people at the first meeting.

Problem of the Week

Problem E

More Surprise Cupcakes

As part of their opening day celebration, Benny's Bakery made 150 cupcakes. The cupcakes all looked identical on the outside, but 6 of them had caramel inside. Customers who bought a cupcake with caramel won a free cake. The cupcakes were randomly arranged and customers were allowed to choose their own cupcakes.

Lin was the first customer in line and bought two cupcakes. Ming was next and also bought two cupcakes. What is the probability that at least one of Ming's cupcakes had caramel inside?

Problem of the Week

Problem E and Solution

More Surprise Cupcakes

Problem

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Lin was the first customer in line and bought two cupcakes. Ming was next and also bought two cupcakes. What is the probability that at least one of Ming's cupcakes had caramel inside?

Solution

To start, we will count the total number of ways to select the first four cupcakes. There are 150 ways to select the first cupcake. For each of these possible selections, there are 149 ways to select the second cupcake. So there are $150 \times 149 = 22\,350$ ways to select the first two cupcakes. For each of these possible selections, there are 148 ways to select the third cupcake. That means there are $22\,350 \times 148 = 3\,307\,800$ ways to select the first three cupcakes. For each of these possible selections, there are a further 147 ways to select the fourth cupcake. So in total, there are $3\,307\,800 \times 147 = 486\,246\,600$ ways to select the first four cupcakes. That's a lot of choices!

Of the 150 available cupcakes, 6 contain caramel and $150 - 6 = 144$ do not. Once a cupcake with caramel is selected, the number of available cupcakes with caramel decreases by 1. Similarly, once a cupcake *without* caramel is selected, the number of available cupcakes *without* caramel decreases by 1.

We now have two choices on how to approach this problem; a direct approach or an indirect approach.

- **Direct Approach:** Determine the total number of ways that Ming can select at least one cupcake with caramel, and then calculate the associated probability.
- **Indirect Approach:** Determine the total number of ways that Ming can select two cupcakes *without* caramel, then calculate the associated probability and subtract this result from 1, because these are complementary events.

We will present three solutions. The first uses an indirect approach, and the second uses a direct approach. The third solution also uses an indirect approach

and gives a much simpler solution to the problem.

Solution 1: Indirect Approach

In how many ways can Ming select two cupcakes *without* caramel? We will look at four cases relating to Lin's cupcakes and calculate the number of ways to select the first four cupcakes for each case.

Case 1: Lin selects two cupcakes without caramel.
The number of possible selections is $144 \times 143 \times 142 \times 141 = 412\,293\,024$.

Case 2: Lin selects a cupcake with caramel and then a cupcake without caramel.
The number of possible selections is $6 \times 144 \times 143 \times 142 = 17\,544\,384$.

Case 3: Lin selects a cupcake without caramel and then a cupcake with caramel.
The number of possible selections is $144 \times 6 \times 143 \times 142 = 17\,544\,384$.

Case 4: Lin selects two cupcakes with caramel.
The number of possible selections is $6 \times 5 \times 144 \times 143 = 617\,760$.

Thus, the total number of ways in which Ming can select two cupcakes without caramel is $412\,293\,024 + 17\,544\,384 + 17\,544\,384 + 617\,760 = 447\,999\,552$. To calculate the probability, we will divide this result by the total number of ways to select four cupcakes.

$$P(\text{Ming selects two cupcakes without caramel}) = \frac{447\,999\,552}{486\,246\,600} = \frac{3432}{3725}$$

The probability of Ming selecting *at least one* cupcake with caramel is equal to 1 minus the probability of Ming selecting two cupcakes *without* caramel.

$$P(\text{Ming selects at least one cupcake with caramel}) = 1 - \frac{3432}{3725} = \frac{293}{3725} \approx 0.079$$

Thus, the probability of Ming selecting at least one cupcake with caramel is $\frac{293}{3725}$, or approximately 8%.

Solution 2: Direct Approach

There are more possibilities to consider when using the direct approach, so we will use a table to show the number of ways for Ming to select at least one cupcake with caramel. Let C represent a cupcake with caramel, and X represent a cupcake without caramel. So CX represents the possibility that the first cupcake contained caramel but the second did not. The possible cases are shown in the table.

Lin's Cupcakes	Ming's Cupcakes	Calculation	Number of Possibilities
CC	CX	$6 \times 5 \times 4 \times 144$	17 280
CC	XC	$6 \times 5 \times 144 \times 4$	17 280
CC	CC	$6 \times 5 \times 4 \times 3$	360
CX	CX	$6 \times 144 \times 5 \times 143$	617 760
CX	XC	$6 \times 144 \times 143 \times 5$	617 760
CX	CC	$6 \times 144 \times 5 \times 4$	17 280
XC	CX	$144 \times 6 \times 5 \times 143$	617 760
XC	XC	$144 \times 6 \times 143 \times 5$	617 760
XC	CC	$144 \times 6 \times 5 \times 4$	17 280
XX	CX	$144 \times 143 \times 6 \times 142$	17 544 384
XX	XC	$144 \times 143 \times 142 \times 6$	17 544 384
XX	CC	$144 \times 143 \times 6 \times 5$	617 760
		Total Possibilities	38 247 048

$$P(\text{Ming selects at least one cupcake with caramel}) = \frac{38\,247\,048}{486\,246\,600} = \frac{293}{3725} \approx 0.079$$

Thus, the probability of Ming selecting at least one cupcake with caramel is $\frac{293}{3725}$, or approximately 8%.

<u>Solution 3</u>: Ignore Lin's Cupcakes

It turns out there is a simpler way to solve this problem if we ignore Lin's cupcakes. We can do this because we don't know how many of Lin's cupcakes had caramel, so removing her cupcakes from the solution does not change the probability that at least one of Ming's cupcakes has caramel. In fact when a customer buys two cupcakes, the probability that at least one cupcake has caramel is the same, regardless of where that customer is in line, as long as we don't know how many cupcakes with caramel have already been purchased.

We will use the indirect approach, so we will first calculate the probability Ming selects two cupcakes without caramel. Since we are ignoring Lin's cupcakes, there are $150 \times 149 = 22\,350$ ways to select the two cupcakes, and $144 \times 143 = 20\,592$ ways to select both cupcakes without caramel. Thus,

$$P(\text{Ming selects two cupcakes } \textit{without} \text{ caramel}) = \frac{20\,592}{22\,350} = \frac{3432}{3725}$$

$$P(\text{Ming selects } \textit{at least one} \text{ cupcake with caramel}) = 1 - \frac{3432}{3725} = \frac{293}{3725} \approx 0.079$$

Thus, the probability of Ming selecting at least one cupcake with caramel is $\frac{293}{3725}$, or approximately 8%.

Problem of the Week

Problem E

Red Dog

We can take any word and rearrange all the letters to get another "word". These new "words" may be nonsensical. For example, you can rearrange the letters in $MATH$ to get $MTHA$.

Nalan wants to rearrange all the letters in $REDDOG$. However, she uses the following rules:

- the letters R, E, and D cannot be adjacent to each other and in that order, and
- the letters D, O, and G cannot be adjacent to each other and in that order.

For example, the "words" $DOGRED$, $DDOGRE$, $GDREDO$, and $DREDOG$ are examples of unacceptable words in this problem, but $DROEGD$ is acceptable.

How many different arrangements of the letters in $REDDOG$ can Nalan make if she follows these rules?

Problem of the Week
Problem E and Solution
Red Dog

Problem

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For example, the "words" $DOGRED$, $DDOGRE$, $GDREDO$, and $DREDOG$ are examples of unacceptable words in this problem, but $DROEGD$ is acceptable.

How many different arrangements of the letters in $REDDOG$ can Nalan make if she follows these rules?

Solution

We will find the total number of possible "words" Nalan can make, and then exclude those "words" which don't follow the rules (i.e. those which contain RED or DOG (or both)).

1. Determine the total number of "words" formed using 2 Ds, 1 E, 1 G, 1 O, and 1 R.

   First, place the E in 6 possible positions. Then, for each of the 6 possible placements of the E, there are 5 ways to place the G. There are then $6 \times 5 = 30$ ways to place the E and the G. Then, for each of the 30 possible placements of the E and G, there are 4 ways to place the O. There are then $30 \times 4 = 120$ ways to place the E, the G, and the O. Then, for each of the 120 possible placements of the E, G, and O, there are 3 ways to place the R. There are then $120 \times 3 = 360$ ways to place the E, the G, the O, and the R.

   For each of the 360 ways to place the E, G, O, and R, the 2 Ds must go in the remaining two empty spaces in 1 way. Therefore, there are $360 \times 1 = 360$ ways to place the E, the G, the O, the R, and the 2 Ds.

   Thus, there are 360 possible "words" that Nalan can make.

2. Determine how many "words" contain RED.

   There are 4 ways to place the word RED in the six spaces. The word RED could start in the first, second, third, or fourth position.

   $$R\ E\ D\ \_\ \_\ \_ \qquad \_\ R\ E\ D\ \_\ \_ \qquad \_\ \_\ R\ E\ D\ \_ \qquad \_\ \_\ \_\ R\ E\ D$$

   For each placement of the word RED, there are 6 ways to place the letters of the word DOG in the remaining three spaces: DOG, DGO, GDO, GOD, ODG and OGD. So there are $4 \times 6 = 24$ "words" containing RED.

3. Determine how many "words" contain DOG.

   There are 4 ways to place the word DOG in the six spaces. The word DOG could start in the first, second, third, or fourth position.

   $$D\ O\ G\ \_\ \_\ \_ \qquad \_\ D\ O\ G\ \_\ \_ \qquad \_\ \_\ D\ O\ G\ \_ \qquad \_\ \_\ \_\ D\ O\ G$$

   For each placement of the word DOG, there are 6 ways to place the letters of the word RED in the remaining three spaces: DER, DRE, EDR, ERD, RDE and RED. So there are $4 \times 6 = 24$ "words" containing DOG.

4. Determine how many "words" contain both RED and DOG.

   There are 4 "words" that contain both RED and DOG. They are as follows.

   $$REDDOG \qquad DOGRED \qquad REDOGD \qquad DREDOG$$

   These 4 "words" have been double-counted, as they would have been counted in both step 2 and step 3.

Thus in total, there are $24 + 24 - 4 = 44$ "words" that contain RED or DOG (or both). Since there are 360 possible "words" Nalan can make, we can subtract 44 from this to determine the number of these "words" that do <u>not</u> contain RED or DOG (or both).

Therefore, $360 - 44 = 316$ "words" can be formed in which the letters R, E and D are not adjacent to each other and in that order and the letters D, O and G are not adjacent to each other and in that order.

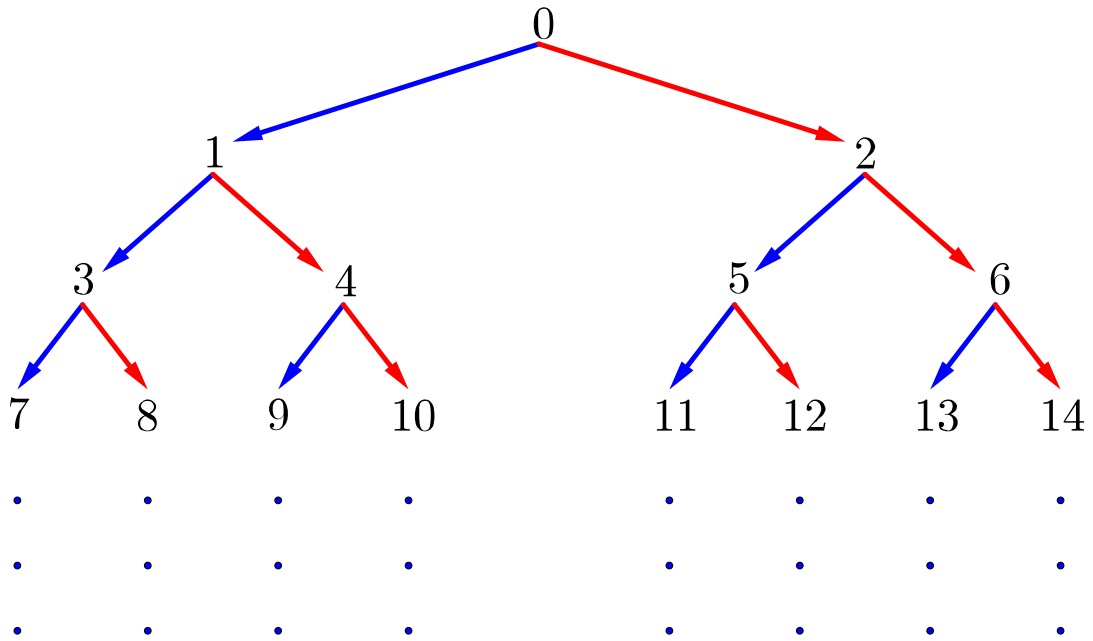
Computational Thinking (C)

Problem of the Week

Problem E

Find the Way

Consider the following number tree.



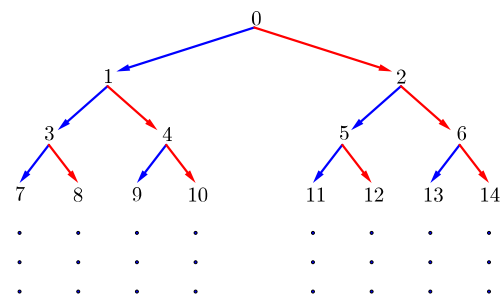
In this number tree, the integers greater than or equal to 0 are written out in increasing order, with the top row containing one integer and every row after containing twice as many integers as the row above it. Each integer is connected to two integers in the row below, one down and to the left and one down and to the right, as shown in the tree. For example, the number 5 is connected to the number 11 (down to the left) and the number 12 (down to the right) in the row below. Notice that we can get from 0 to 12 by going down right (R), down left (L), then down right (R).

What is the sequence of left (L) and right (R) movements to get from the number 0 to the number 1172 in the tree?

Problem of the Week
Problem E and Solution
Find the Way

Problem

Consider the number tree shown to the right. In this number tree, the integers greater than or equal to 0 are written out in increasing order, with the top row containing one integer and every row after containing twice as many integers as the row above it. Each integer is connected to two integers in the row below, one down and to the left and one down and to the right, as shown in the tree. For example, the number 5 is connected to the number 11 (down to the left) and the number 12 (down to the right) in the row below. Notice that we can get from 0 to 12 by going down right (R), down left (L), then down right (R).



What is the sequence of left (L) and right (R) movements to get from the number 0 to the number 1172 in the tree?

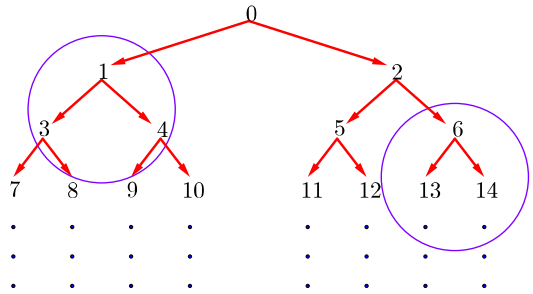
Solution

Solution 1

To begin, we will make an observation concerning the tree. When we perform a move to the left (L) from any number, we end up at an odd number. When we perform a move to the right (R) from any number, we end up at an even number. So the final move to get to the number 1172 was a move to the right (R) since 1172 is an even number. Is there a general formula which can be used to determine the number you end up at when asked to move right (R)? Is there a general formula which can be used when asked to move left (L)?

The diagram below has two parts of the tree circled. Can we discover a pattern that takes us from each initial number to the odd and even numbers below? To get from 1 to 3 we could add 2 and to get from 1 to 4 we could add 3. But doing this from 6 would not get us to 13 and 14.

As we go down the tree, each new row has twice as many numbers as the row above. Let's try multiplying the initial number by 2 and then seeing what is necessary to get to the odd and even number below. If we double 1, we get 2. Then we would need to add 1 to get to the odd number 3 below and add 2 to get to the even number 4 below. Does this work with the 6? If we double 6 and add 1, we get 13. It appears to work. If we double 6 and add 2, we get 14. It also appears to work.



So it would appear that if we make a move left (L) from any number a in the tree, the resulting number is one more than twice the value of a. That is, a move left (L) from a takes us to the number $2a + 1$ in the tree.

It would appear that if we make a move right (R) from any number a in the tree, the resulting number is two more than twice the value of a. That is, a move right (R) from a takes us to the number $2a + 2$ in the tree.

The results are true but unproven. This relationship has worked for all of the rows we have sampled but we have not proven it true in general. You will have to wait for some higher mathematics to be able to prove that this is true in general.

To go back up the tree, we could determine an inverse move to undo a move to the left (L) or to the right (R).

A move to the left (L) takes a to an odd number n such that $n = 2a + 1$. Solving this equation for a we get $a = \frac{n-1}{2}$. A move to the right (R) takes a to an even number n such that $n = 2a + 2$. Solving this equation for a we get $a = \frac{n-2}{2}$.
We can now move from 1172 up the tree to 0 using the appropriate inverse formula each time.

Initial Number	Odd or Even	Calculation	Previous Number
1172	even	$\frac{1172-2}{2}$	585
585	odd	$\frac{585-1}{2}$	292
292	even	$\frac{292-2}{2}$	145
145	odd	$\frac{145-1}{2}$	72
72	even	$\frac{72-2}{2}$	35
35	odd	$\frac{35-1}{2}$	17
17	odd	$\frac{17-1}{2}$	8
8	even	$\frac{8-2}{2}$	3
3	odd	$\frac{3-1}{2}$	1
1	odd	$\frac{1-1}{2}$	0

From our work in the table above, we can see that the path to 1172 goes through the following numbers:

$$0 \to 1 \to 3 \to 8 \to 17 \to 35 \to 72 \to 145 \to 292 \to 585 \to 1172$$

By looking at each successive number in the sequence in terms of its parity (odd or even) we can determine the required sequence of moves:

$$\begin{array}{ccccccccccc} & \text{L} & \text{L} & \text{R} & \text{L} & \text{L} & \text{R} & \text{L} & \text{R} & \text{L} & \text{R} \\ 0 \to & 1 \to & 3 \to & 8 \to & 17 \to & 35 \to & 72 \to & 145 \to & 292 \to & 585 \to & 1172 \end{array}$$

A solution that does not require the use of the unproven result is provided on the next page.

Solution 2

Which row of the tree contains the number 1172?

To get from the top integer to the rightmost integer in row 2, add 2. To get from the rightmost integer in row 2 to the rightmost integer in row 3, add 4. To get from the rightmost integer in row 3 to the rightmost integer in row 4, add 8. These numbers which are added correspond to the number of integers in the next row. Working through this we can discover that 1172 is in the 11th row.

The rightmost integer in row 10 is: $0 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 = 1022$.
The rightmost integer in row 11 is: $0 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 = 2046$.
It follows that the number 1172 is in row 11 which contains the 1024 integers ranging from 1023 (1 more than the last integer in row 10) to 2046 (the rightmost integer in row 11), inclusive.

Now we look at the commands to narrow down where we will be in the 11th row.

Row 11 can be broken into two halves, the left half contains 1023 to 1534 and the right half contains 1535 to 2046. The number 1172 is in the left half, so we needed to go left (L) first from row 1 to row 2.

Row 11 from 1023 to 1534 can be broken into two halves, the left half contains 1023 to 1278 and the right half contains 1279 to 1534. The number 1172 is in the left half, so we needed to go left (L) from row 2 to row 3.

Row 11 from 1023 to 1278 can be broken into two halves, the left half contains 1023 to 1150 and the right half contains 1151 to 1278. The number 1172 is in the right half, so we needed to go right (R) from row 3 to row 4.

Row 11 from 1151 to 1278 can be broken into two halves, the left half contains 1151 to 1214 and the right half contains 1215 to 1278. The number 1172 is in the left half, so we needed to go left (L) from row 4 to row 5.

Row 11 from 1151 to 1214 can be broken into two halves, the left half contains 1151 to 1182 and the right half contains 1183 to 1214. The number 1172 is in the left half, so we needed to go left (L) from row 5 to row 6.

Row 11 from 1151 to 1182 can be broken into two halves, the left half contains 1151 to 1166 and the right half contains 1167 to 1182. The number 1172 is in the right half, so we needed to go right (R) from row 6 to row 7.

Row 11 from 1167 to 1182 can be broken into two halves, the left half contains 1167 to 1174 and the right half contains 1175 to 1182. The number 1172 is in the left half, so we needed to go left (L) from row 7 to row 8.

Row 11 from 1167 to 1174 can be broken into two halves, the left half contains 1167 to 1170 and the right half contains 1171 to 1174. The number 1172 is in the right half, so we needed to go right (R) from row 8 to row 9.

Row 11 from 1171 to 1174 can be broken into two halves, the left half contains 1171 to 1172 and the right half contains 1173 to 1174. The number 1172 is in the left half, so we needed to go left (L) from row 9 to row 10.

And finally, row 11 from 1171 to 1172 can be broken into two halves, the left half contains 1171 and the right half contains 1172. The number 1172 is in the right half, so we needed to go right (R) from row 10 to row 11.

Therefore, we get to 1172 using the following sequence of movements:

$$\mathrm{L} \to \mathrm{L} \to \mathrm{R} \to \mathrm{L} \to \mathrm{L} \to \mathrm{R} \to \mathrm{L} \to \mathrm{R} \to \mathrm{L} \to \mathrm{R}$$

Problem of the Week

Problem E

Best in Show

Four puppies, Bruno, Kitty, O'Reilly, and Sweetie, competed in the Annual Puppies of the World Fair. Prizes were awarded for the top three competitors as follows: first place received a Red Ribbon, second place received a Yellow Ribbon, and third place received a Blue Ribbon.

Three members of the audience predicted how the prizes would be awarded.

- Lucie predicted that Sweetie would win a Red Ribbon, Kitty would win a Yellow Ribbon, and O'Reilly would win a Blue Ribbon.
- Jenn predicted that Bruno would win a Red Ribbon, Sweetie would win a Yellow Ribbon, and O'Reilly would win a Blue Ribbon.
- Brenton predicted that Kitty would win a Red Ribbon, Bruno would win a Yellow Ribbon, and Sweetie would win a Blue Ribbon.

It turns out that each audience member predicted exactly one prize winner correctly.

Determine which dog won which prize.

Problem of the Week

Problem E and Solution

Best in Show

Problem

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- Jenn predicted that Bruno would win a Red Ribbon, Sweetie would win a Yellow Ribbon, and O'Reilly would win a Blue Ribbon.
- Brenton predicted that Kitty would win a Red Ribbon, Bruno would win a Yellow Ribbon, and Sweetie would win a Blue Ribbon.

It turns out that each audience member predicted exactly one prize winner correctly.

Determine which dog won which prize.

Solution

Let's look at Lucie's prize predictions. Assume she is correct that Sweetie won a Red Ribbon. This leads to the following six possibilities.

First - Red Ribbon	Second - Yellow Ribbon	Third - Blue Ribbon	Fourth - No Prize	
Sweetie	Bruno	Kitty	O'Reilly	(1)
Sweetie	Bruno	O'Reilly	Kitty	(2)
Sweetie	Kitty	Bruno	O'Reilly	(3)
Sweetie	Kitty	O'Reilly	Bruno	(4)
Sweetie	O'Reilly	Bruno	Kitty	(5)
Sweetie	O'Reilly	Kitty	Bruno	(6)

Since the other two parts of Lucie's prize prediction are not correct, Kitty cannot win a Yellow Ribbon prize so we can rule out (3) and (4), and O'Reilly cannot win a Blue Ribbon so we can also rule out (2). This leaves (1), (5) and (6) as the only possibilities for Lucie.

First - Red Ribbon	Second - Yellow Ribbon	Third - Blue Ribbon	Fourth - No Prize	
Sweetie	Bruno	Kitty	O'Reilly	(1)
Sweetie	O'Reilly	Bruno	Kitty	(5)
Sweetie	O'Reilly	Kitty	Bruno	(6)

Now look at Jenn's prize predictions in light of Lucie's three valid possibilities. Jenn makes exactly one true prize prediction. None of Lucie's possibilities would ever make one of Jenn's prize predictions true. Therefore, our original assumption that Lucie correctly predicted Sweetie to win a Red Ribbon was incorrect.

So now we assume that Lucie correctly predicts that Kitty would win a Yellow Ribbon prize. This leads to the following six possibilities.

First - Red Ribbon	Second - Yellow Ribbon	Third - Blue Ribbon	Fourth - No Prize	
Bruno	Kitty	O'Reilly	Sweetie	(7)
Bruno	Kitty	Sweetie	O'Reilly	(8)
O'Reilly	Kitty	Bruno	Sweetie	(9)
O'Reilly	Kitty	Sweetie	Bruno	(10)
Sweetie	Kitty	Bruno	O'Reilly	(11)
Sweetie	Kitty	O'Reilly	Bruno	(12)

Since the other two parts of Lucie's prize prediction are not correct, Sweetie cannot win a Red Ribbon so we can rule out (11) and (12), and O'Reilly cannot win a Blue Ribbon so we can also rule out (7). This leaves (8), (9) and (10) as the only possibilities for Lucie.

First - Red Ribbon	Second - Yellow Ribbon	Third - Blue Ribbon	Fourth - No Prize	
Bruno	Kitty	Sweetie	O'Reilly	(8)
O'Reilly	Kitty	Bruno	Sweetie	(9)
O'Reilly	Kitty	Sweetie	Bruno	(10)

Now look at Jenn's prize predictions in light of Lucie's three valid possibilities. Jenn makes exactly one true prize prediction. Possibility (8) is the only one of the possibilities that works for Jenn.
Now we must check Brenton's prize prediction to see if it is still valid with (8). Brenton predicted that Kitty would win a Red Ribbon, Bruno would win a Yellow Ribbon and Sweetie would win a Blue Ribbon. This prediction works since exactly one of Brenton's prize predictions, that Sweetie wins a Blue Ribbon, is true. The other two predictions are false.

We should check to see that this is the only correct solution. We do this by assuming Lucie's third prediction, O'Reilly wins a Blue Ribbon, was her only correct prediction. This leads to the following six possibilities.

First - Red Ribbon	Second - Yellow Ribbon	Third - Blue Ribbon	Fourth - No Prize	
Bruno	Kitty	O'Reilly	Sweetie	(13)
Bruno	Sweetie	O'Reilly	Kitty	(14)
Kitty	Bruno	O'Reilly	Sweetie	(15)
Kitty	Sweetie	O'Reilly	Bruno	(16)
Sweetie	Bruno	O'Reilly	Kitty	(17)
Sweetie	Kitty	O'Reilly	Bruno	(18)

Since the other two parts of Lucie's prize prediction are not correct, Sweetie cannot win a Red Ribbon so we can rule out (17) and (18), and Kitty cannot win a Yellow Ribbon so we can also rule out (13). This leaves (14), (15) and (16) as the only possibilities for Lucie.

First - Red Ribbon	Second - Yellow Ribbon	Third - Blue Ribbon	Fourth - No Prize	
Bruno	Sweetie	O'Reilly	Kitty	(14)
Kitty	Bruno	O'Reilly	Sweetie	(15)
Kitty	Sweetie	O'Reilly	Bruno	(16)

Now look at Jenn's prize predictions in light of Lucie's three valid possibilities. Jenn makes exactly one true prize prediction. Possibility (15) is the only one of the possibilities that works for Jenn. In both (14) and (16) Jenn would make at least two correct prize predictions.
Now we must check Brenton's prize prediction to see if it is still valid with (15). Brenton predicted that Kitty would win a Red Ribbon, Bruno would win a Yellow Ribbon and Sweetie would win a Blue Ribbon. Since two of Brenton's prize predictions would be true in (15) but Brenton only made one correct prize prediction, (15) does not work for Brenton.

Therefore, the only possibility is that Bruno finishes first and wins a Red Ribbon, Kitty finishes second and wins a Yellow Ribbon, and Sweetie finishes third and gets a Blue Ribbon. O'Reilly finishes fourth.

Problem of the Week
Problem E
Is It Valid?

Debit and credit cards contain account numbers which consist of many digits. When purchasing items online, you are often asked to type in your account number. Because there are so many digits, it is easy to type the number incorrectly. The last digit of the number is a specially generated check digit which can be used to quickly verify the validity of the number. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid.

The steps performed in the Luhn Algorithm are outlined in the flowchart below. Two examples are provided.

Example 1:

Number: 135792

Reversal: 297531

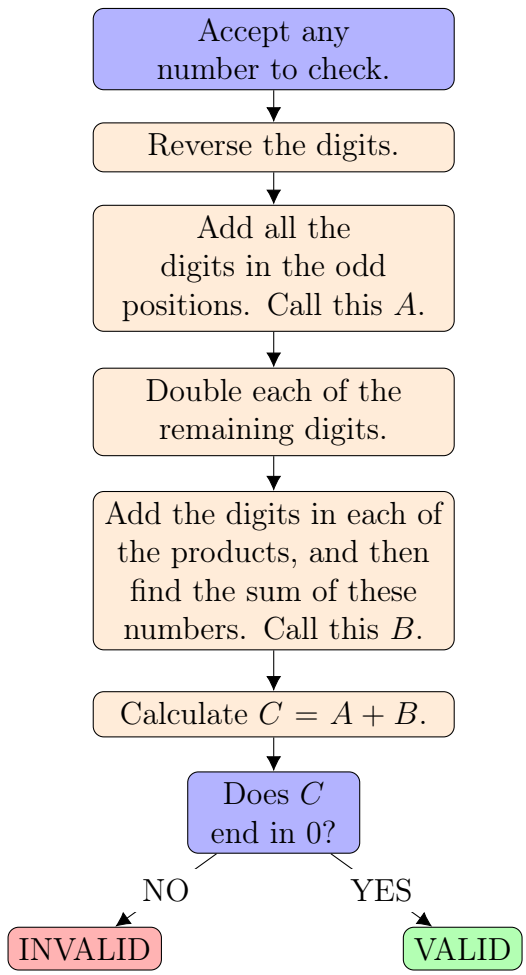
$A = 2 + 7 + 3$
$= 12$

$2 \times 9 = 18$
$2 \times 5 = 10$
$2 \times 1 = 2$

$B = (1 + 8) + (1 + 0) + 2$
$= 9 + 1 + 2$
$= 12$

$C = 12 + 12 = 24$

C does not end in zero.
The number is not valid.



Example 2:

Number: 1357987

Reversal: 7897531

$A = 7 + 9 + 5 + 1$
$= 22$

$2 \times 8 = 16$
$2 \times 7 = 14$
$2 \times 3 = 6$

$B = (1 + 6) + (1 + 4) + 6$
$= 7 + 5 + 6$
$= 18$

$C = 22 + 18 = 40$

C ends in zero.
The number is valid.

Suppose the number 4633 $RT0R$ 481 is a valid number when verified by the Luhn Algorithm, where R and T are each integers from 0 to 9 such that $R \leq T$.

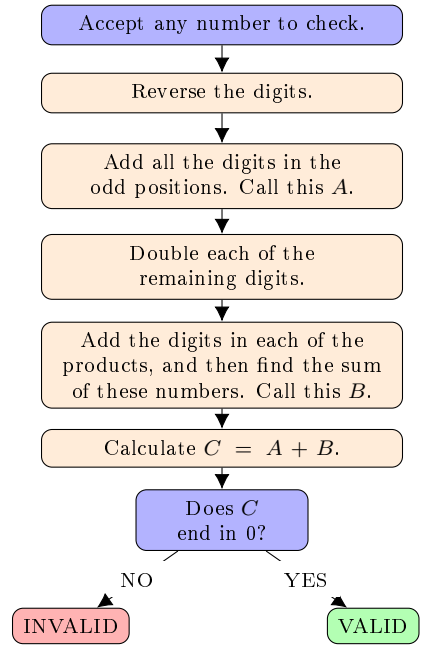
Determine all possible values of R and T.

Problem of the Week
Problem E and Solution
Is It Valid?

Problem

Debit and credit cards contain account numbers which consist of many digits. When purchasing items online, you are often asked to type in your account number. Because there are so many digits, it is easy to type the number incorrectly. The last digit of the number is a specially generated check digit which can be used to quickly verify the validity of the number. A common algorithm used for verifying numbers is called the *Luhn Algorithm*. A series of operations are performed on the number and a final result is produced. If the final result ends in zero, the number is valid. Otherwise, the number is invalid.

The steps performed in the Luhn Algorithm are outlined in the flowchart to the right.



Suppose the number 4633 $RT0R$ 481 is a valid number when verified by the Luhn Algorithm, where R and T are each integers from 0 to 9 such that $R \leq T$.

Determine all possible values of R and T.

Solution

When the digits of the number are reversed the resulting number is 184 $R0TR$ 3364. The sum of the digits in the odd positions is $A = 1 + 4 + 0 + R + 3 + 4 = 12 + R$.

When the digits in the remaining positions are doubled, the following products are obtained:

$$2 \times 8 = 16;\ 2 \times R = 2R;\ 2 \times T = 2T;\ 2 \times 3 = 6;\ \text{and } 2 \times 6 = 12$$

Let x represent the sum of the digits of $2R$ and y represent the sum of the digits of $2T$. When the digit sums from each of the products are added, the sum is:

$$B = (1 + 6) + x + y + 6 + (1 + 2) = 7 + x + y + 6 + 3 = x + y + 16$$

C is the sum of A and B, so $C = 12 + R + x + y + 16 = 28 + R + x + y$.

When an integer from 0 to 9 is doubled and the digits of the product are added together, what are the possible sums which can be obtained?

Original Digit (R or T)	0	1	2	3	4	5	6	7	8	9
Twice the Original Digit ($2R$ or $2T$)	0	2	4	6	8	10	12	14	16	18
The Sum of the Digits of $2R$ or $2T$ (x or y)	0	2	4	6	8	1	3	5	7	9

Notice that the sum of the digits of twice the original digit can only be an integer from 0 to 9 inclusive. It follows that the only values for x or y are the integers from 0 to 9.

The first row and the last row of this chart will be reprinted at the top of the next page as the results are used in the solution. We will use it to determine x from R and to determine T from y.

Original Digit (R or T)	0	1	2	3	4	5	6	7	8	9
The Sum of the Digits of $2R$ or $2T$ (x or y)	0	2	4	6	8	1	3	5	7	9

Since the number is valid, $C = 28 + R + x + y$ must end in zero.

What are the possible values to consider for C? Since the maximum value for each of R, x, and y is 9, the maximum value for $R + x + y$ is 27 and the maximum value for $C = 28 + R + x + y$ is 55. It follows that the only valid possibilities for C that end in zero are 30, 40, and 50. We will consider each of the three possibilities.

1. $C = 30$ and $R + x + y = 2$

   If $R = 0$, then $x = 0$, $y = 2$ and $T = 1$.
   There are no other valid possibilities for R so that $R + x + y = 2$.

   Therefore, this case produces one valid possibility for (R, T): $(0, 1)$.

2. $C = 40$ and $R + x + y = 12$

R	x	$y = 12 - R - x$	T	Valid or Invalid	(R, T)
0	0	12		not valid, $y > 9$	
1	2	9	9	valid, $R \leq T$	$(1, 9)$
2	4	6	3	valid, $R \leq T$	$(2, 3)$
3	6	3	6	valid, $R \leq T$	$(3, 6)$
4	8	0	0	not valid, $R > T$	
5	1	6	3	not valid, $R > T$	
6	3	3	6	valid, $R \leq T$	$(6, 6)$
7	5	0	0	not valid, $R > T$	
8	7	-3		not valid, $y < 0$	
9	9	-6		not valid, $y < 0$	

   Therefore, this case produces four valid possibilities for (R, T): $(1, 9)$, $(2, 3)$, $(3, 6)$ and $(6, 6)$.

3. $C = 50$ and $R + x + y = 22$

R	x	$y = 22 - R - x$	T	Valid or Invalid	(R, T)
0	0	22		not valid, $y > 9$	
1	2	19		not valid, $y > 9$	
2	4	16		not valid, $y > 9$	
3	6	13		not valid, $y > 9$	
4	8	10		not valid, $y > 9$	
5	1	16		not valid, $y > 9$	
6	3	13		not valid, $y > 9$	
7	5	10		not valid, $y > 9$	
8	7	7	8	valid, $R \leq T$	$(8, 8)$
9	9	4	2	not valid, $R > T$	

   Therefore, this case produces one valid possibility for (R, T): $(8, 8)$.

We have examined all possible values for C.

Therefore, there are six valid possibilities for (R, T): $(0, 1)$, $(1, 9)$, $(2, 3)$, $(3, 6)$, $(6, 6)$ and $(8, 8)$.

These correspond to the following six card numbers, which are indeed valid by the Luhn Algorithm: 4633 0100 481, 4633 1901 481, 4633 2302 481, 4633 3603 481, 4633 6606 481, and 4633 8808 481.