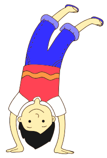


Problem of the Week
Problem A and Solution
Flipping for Fun

Problem

James decides to do an experiment. He flips a coin to determine how he moves. If the coin shows heads, then he moves one step to the right. If the coin shows tails, then he takes one step to the left. Assuming that the coin he is using is fair, (in other words it has an equal chance of landing on heads or tails), answer the following questions:

A) What is the probability that James will be back where he started after flipping the coin twice?

B) What is the probability that James will be within one step of where he started after flipping the coin three times?

C) What is the probability that James will be back where he started after flipping the coin four times?

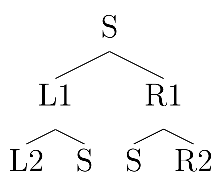
D) What is the probability that James will be back where he started after flipping the coin five times?

Solution

One way to keep track of James' movement, is to make a diagram. James will start on the square labelled **S**, and may move to one of the squares after flipping the coin.

L5	**L4**	**L3**	**L2**	**L1**	**S**	**R1**	**R2**	**R3**	**R4**	**R5**

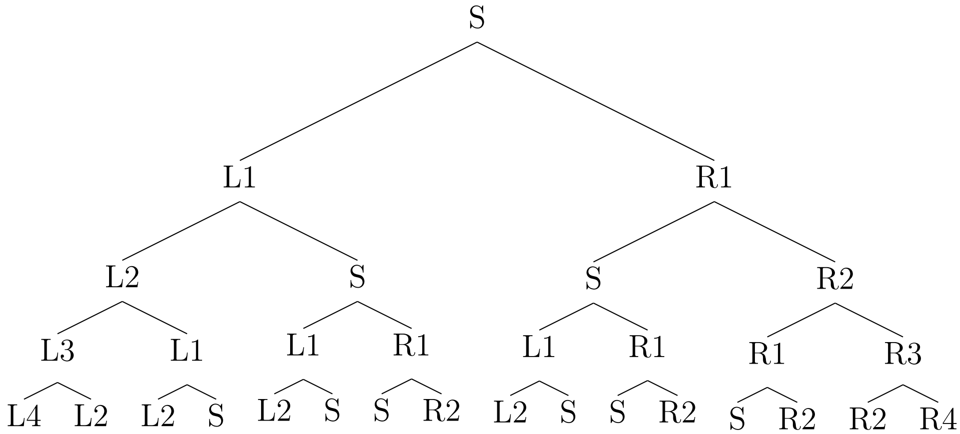
We can use a tree to show all the possibilities of where James ends up after flipping the coin. If the coin shows heads after flipping, James' position is shown by following a right branch in the tree; if the coin shows tails after flipping, James' position is shown by following the left branch in the tree. So after flipping the coin twice this tree shows James' possible positions:



For example, if the first time he flipped the coin it landed on tails, we follow the left branch of the tree and see that James is at position **L1** From there, if he flipped the coin and it also landed on tails, we follow the left branch of the tree and see that James is at position **L2**.

A) All four of the outcomes at the bottom of the tree are equally likely when flipping a fair coin. Since two out of the four outcomes result in James ending up at the starting position, the probability that he is back where he started is 2 out of 4 which is equal to 1 out of 2.

We can continue tracking where James might end up by adding to the tree. Each level in the tree shows James' possible positions after another coin flip.



The bottom row of the tree shows James' possible positions after flipping the coin four times. The second from the bottom row shows James' possible positions after flipping the coin three times. Each outcome in a row is equally likely.

B) After three flips, James is expected be at position **L1** 3 out of 8 times, and he is expected to be at position **R1** 3 out of 8 times. So a total of 6 out of 8 times, James is expected to be within one step of where he started. The probability he is within one step of his starting position is 6 out of 8 which is equal to 3 out of 4.

   Alternatively, if we did not have the tree diagram, we could simply list the possible sequences of three coin flips: `(H H H)`, `(H H T)`, `(H T H)`, `(H T T)`, `(T H H)`, `(T H T)`, `(T T H)`, and `(T T T)`. In six out of eight of these sequences, James ends up one step away from where he started.

C) From the information on the bottom row, of the tree, we see that the probability that James is back at the starting position is 6 out of 16 which is equal to 3 out of 8.

D) We could continue adding one more level to the tree to see how many outcomes are equal to **S** after flipping the coin five times. However, looking at the bottom row of the tree, we can observe that all of those outcomes are either **S** or at least two steps away from **S**. If James flips the coin once, and he is standing at the starting position, then he cannot end up at the starting position. If he is at least two steps away from the starting position, then he cannot return to the starting position after just one coin flip. So, there is no way that James can end up at the starting position after flipping the coin five times. The probability is 0.

   Another way to look at this problem is to notice that if James flips a coin an odd number of times, James always ends up on an odd numbered square. Every time James flips the coin an even number of times, he ends up on either an even numbered square, or on the **S**. So after five flips, James must land on an odd numbered square, so there is no chance that James could land on the starting point.

Teacher's Notes

This problem is actually a demonstration of a statistics concept known as a *random walk*. This is a concept that is introduced in upper year university or college courses. You can try to predict future events, by looking at what might happen if you make random choices. This concept is used to predict share prices in economics, to describe how the molecules of liquids and gasses move, and to help automate image recognition of digital images, among other applications.

Keeping track of all of the details of the possible walks that James takes, can be difficult. However, if we look at the problem one step at a time, we can figure out how many walks there could be. If James flips the coin once, there are two possible paths James could take. From each of those two paths, if we flip a coin, then there are two possible paths James could take. So after two flips we have a total of four paths. After each flip, we double the number of possible paths James could take. So after three flips, there are eight possible paths, and after four flips, there are 16 possible paths.

So rather than listing all of the possible paths, James could focus on listing the ways that he could flip a coin to get a to particular destination. For example, in part C) of the problem, James ends up at the starting position, when he flips the coins and it lands on heads and tails an equal number of times. In particular he ends up at the starting spot after these flips:

```
H T H T
H T T H
H H T T
T H T H
T H H T
T T H H
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