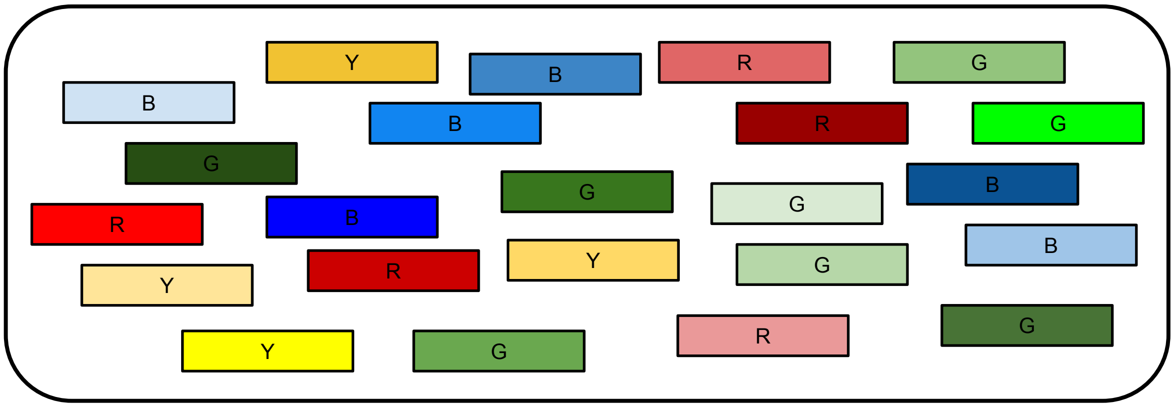
Problem of the Week
Problem A and Solution
Colourful Conundrum

Problem

Ian has a pencil case full of coloured pencils. He has many shades of four main colours: red (R), blue (B), green (G), and yellow (Y). The diagram below represents the contents of the pencil case:



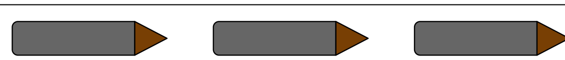
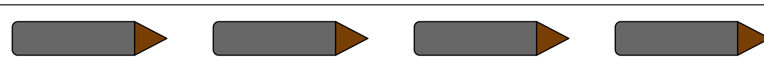
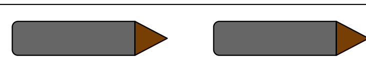
A) Use the chart below to tally the coloured pencils and then create a pictograph using the key provided:

B) How many pencils will you have to draw from the pencil case to guarantee you will get two pencils of the same main colour?

Solution

A) Here is the completed tally and pictograph for the pencil case data:

Key:  = 2 pencils

Main Colour	Tally	Pictograph
red	𝍸	
blue	𝍸 𝍷	
green	𝍸 𝍷𝍷𝍷	
yellow	𝍷𝍷𝍷𝍷	

B) Since there are four main colours, then it is possible to pick four pencils from the case that all have different main colours. So to guarantee you will get two pencils of the same main colour, you will need to pick more than four from the case. Since there are not five main colours, if you pick five pencils, at least two of the pencils will be the same colour. So five is the smallest number of pencils you could draw that would guarantee two of the same main colour.

Teacher's Notes

The solution for part **B** is an example of the *Pigeonhole Principle* at work. The following analogy is often used to describe the Pigeonhole Principle.

> Imagine that you have n pigeons and that you have k pigeonholes where the birds roost. All of the pigeons come home to roost at night. If $n > k$ (i. e. the number of pigeons is greater than the number of pigeonholes), then at least one of the pigeonholes contains more than one pigeon.
>
> We can show that this is true by trying to prove the opposite. Let's assume that no pigeonhole contains more than one pigeon. As each pigeon flies home to roost, it enters an empty pigeon hole. However, after k pigeons have come home all of the pigeonholes will contain a bird. If $n > k$, then there is at least one more pigeon that needs somewhere to roost. It will need to share one of the pigeonholes.

Note that the Pigeonhole Principle does not guarantee that each pigeonhole contains a pigeon. In theory (although maybe not in reality), all of the pigeons could be roosting in one pigeonhole and there would be many empty spots. The only guarantee is that there will be at least one pigeonhole that contains more than one pigeon.

The key to using the Pigeonhole Principle in a mathematics problem is to determine what are the *pigeons* and what are the *pigeonholes*. In part **B** of this problem we can imagine that we have one container, each of which is labelled with one of the four main colours. As we draw pencils out of the case, we will put them in the container labelled with a matching main colour. The question becomes, how many pencils do we need to draw from the pencil case before at least one of the containers has two pencils? In this case, the containers are like the pigeonholes and the coloured pencils are like the pigeons. Since there are four containers, then if we have more than four coloured pencils, one of the containers must have at least two pencils. In particular, when we draw five pencils from the case we must have two that are the same main colour. Since five is the smallest integer that is greater than four, then five is the fewest number of pencils we must draw to guarantee two have the same main colour. However, we do not know which two are the same main colour, and it is possible we have three, four, or even all five pencils that are the same main colour.