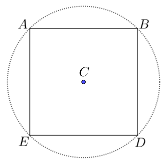
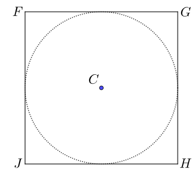



# Problem of the Week
## Problem C and Solution
## Pi Day Squares



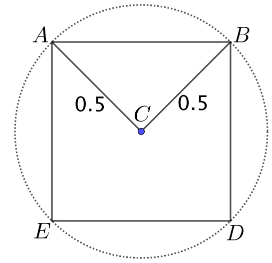

### Problem

Archimedes determined lower bounds (minimum values) for $\pi$ by finding the perimeters of regular polygons inscribed in a circle with diameter of length 1. He also determined upper bounds (maximum values) for $\pi$ by finding the perimeters of regular polygons circumscribed in a circle with diameter of length 1. We will determine such bounds by looking at squares inscribed and circumscribed in a circle with centre $C$ and diameter 1. Since the circle has circumference equal to $\pi$, the perimeter of the inscribed square $ABDE$ will give a lower bound for $\pi$ and the perimeter of the circumscribed square $FGHJ$ will give an upper bound for $\pi$. Using these squares, determine a lower bound and an upper bound for $\pi$.

### Solution

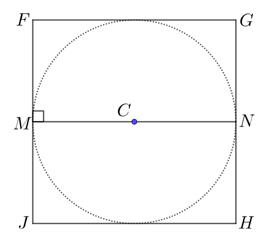
For the inscribed square, draw line segments $AC$ and $BC$. Both $AC$ and $BC$ are radii of the circle with diameter 1 so $AC = BC = 0.5$. Since the diagonals of a square are perpendicular at $C$, it follows that $\triangle ACB$ is a right triangle with $\angle ACB = 90°$. We can use the Pythagorean Theorem to find the length of $AB$.

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= (0.5)^2 + (0.5)^2 \\ &= 0.25 + 0.25 \\ &= 0.5 \\ AB &= \sqrt{0.5} \text{ (since } AB > 0) \\ &\approx 0.707 \end{aligned}$$




Since $AB$ is one of the sides of the inscribed square, the perimeter of square $ABDE$ is equal to $4 \times AB \approx 4 \times 0.707 = 2.828$. This gives us a lower bound for $\pi$.

For the circumscribed square, let $M$ be the point of tangency on side $FJ$ and let $N$ be the point of tangency on $GH$. Draw radii $CM$ and $CN$. Since $M$ is a point of tangency, we know that $\angle FMC = 90°$, and thus $CM$ is parallel to $FG$. Similarly, $CN$ is parallel to $FG$.




Thus, $MN$ is a straight line segment, and since it passes through $C$, the centre of the circle, $MN$ must also be a diameter of the circle. Thus, $MN = 1$. Also, $FMNG$ is a rectangle, so $FG = MN = 1$ and the perimeter of square $FGHJ$ is equal to $4 \times FG = 4(1) = 4$. This is an upper bound for $\pi$.

Therefore, a lower bound (minimum value) for $\pi$ is approximately 2.828 and an upper bound (maximum value) for $\pi$ is 4. That is, $2.828 < \pi < 4$.

**NOTE:** Since we know that $\pi \approx 3.14$, these are not the best bounds for $\pi$. Archimedes used regular polygons with more sides to get better approximations. In the POTW D problem, we investigate using regular hexagons to get a better approximation.