Problem of the Week
Problem C and Solution
No Lemons, No Melon

Problem

A palindrome is a word, phrase, sentence, or number that reads the same forwards and backwards. Determine the smallest and largest seven-digit palindromic numbers that are divisible by 15.

Solution

We are looking for two seven-digit numbers of the form $abcdcba$.

For a number to be divisible by 15, it must be divisible by both 3 and 5.

To be divisible by 5, a number must end in 0 or 5. If the required number ends in 0, it must also begin with 0 in order to be a palindrome. But the number $0bcdcb0$ is not a seven-digit number, since the leading digit cannot be 0. Therefore, the number cannot end in a 0 and hence must start and end with a 5. The required numbers look like $5bcdcb5$.

For a number to be divisible by 3, the sum of its digits must be divisible by 3.

For the smallest possible number, let $b = 0$ and $c = 0$ in $5bcdcb5$. We must find the smallest value of d so that $500d005$ is divisible by 3. The sum of the digits is $5 + 0 + 0 + d + 0 + 0 + 5 = d + 10$ and d can take on any integer value from 0 to 9. It follows that $d + 10$ can take on integer values from 10 to 19. The smallest number in this range divisible by 3 is 12 so $d + 10 = 12$ and $d = 2$.

For the largest possible number, let $b = 9$ and $c = 9$ in $5bcdcb5$. We must find the largest value of d so that $599d995$ is divisible by 3. The sum of the digits is $5 + 9 + 9 + d + 9 + 9 + 5 = d + 46$ and d can take on any integer value from 0 to 9. It follows that $d + 46$ can take on integer values from 46 to 55. The largest number in this range divisible by 3 is 54 so $d + 46 = 54$ and $d = 8$.

The smallest and largest seven-digit palindromic numbers exactly divisible by 15 are 5002005 and 5998995, respectively.

Extension: How many seven-digit palindromes of the form $50cdc05$ are divisible by 15?