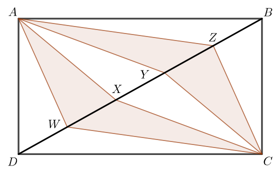

Problem of the Week

Problem D and Solution

A Shade Smaller

Problem

The diagonal, BD, of rectangle $ABCD$ is divided into 5 equal segments at W, X, Y, and Z. If rectangle $ABCD$ has length $AB = 9$ and width $AD = 5$, determine the area of the shaded region.

Solution

Solution 1

Using the formula for area of a triangle $= \frac{\text{base}\times\text{height}}{2}$, we have area $\triangle ABD = \frac{9\times5}{2} = \frac{45}{2}$ units^2.

The triangles $\triangle DAW, \triangle WAX, \triangle XAY, \triangle YAZ$ and $\triangle ZAB$ have the same height. Since $DW = WX = XY = YZ = ZB$, the triangles also have equal bases. Therefore,
area $\triangle DAW$ = area $\triangle WAX$ = area $\triangle XAY$ = area $\triangle YAZ$ = area $\triangle ZAB$
$= \frac{1}{5}$ (area $\triangle ABD$) $= \frac{1}{5}\left(\frac{45}{2}\right) = \frac{9}{2}$ units^2.

Similarly, the area of $\triangle BCD$ is $\frac{9\times5}{2} = \frac{45}{2}$ units^2.

The triangles $\triangle DCW, \triangle WCX, \triangle XCY, \triangle YCZ$ and $\triangle ZCB$ have the same height and equal bases. Therefore, area $\triangle DCW$ = area $\triangle WCX$ = area $\triangle XCY$ = area $\triangle YCZ$ = area $\triangle ZCB$
$= \frac{1}{5}$ (area $\triangle BCD$) $= \frac{1}{5}\left(\frac{45}{2}\right) = \frac{9}{2}$ units^2.

Therefore, the area of the shaded region is $4\left(\frac{9}{2}\right) = 18$ units^2.

Solution 2

Since $ABCD$ is a rectangle, the angle at A is 90°. We can then use the Pythagorean Theorem to calculate $BD^2 = AB^2 + AD^2 = 9^2 + 5^2 = 81 + 25 = 106$, and so $BD = \sqrt{106}$, since $BD > 0$. Therefore, $DW = WX = XY = YZ = ZB = \frac{1}{5}(BD) = \frac{1}{5}\sqrt{106}$.

Using the formula area of a triangle $= \frac{\text{base}\times\text{height}}{2}$, base $AB = 9$ and height $AD = 5$, we can calculate area $\triangle ABD = \frac{9\times5}{2} = \frac{45}{2}$ units^2.

Let's treat $BD = \sqrt{106}$ as the base of $\triangle ABD$ and let h be the corresponding height. Since the area of $\triangle ABD$ is $\frac{45}{2}$, then we have $\frac{\sqrt{106}\times h}{2} = \frac{45}{2}$ and so $\sqrt{106}\times h = 45$, thus $h = \frac{45}{\sqrt{106}}$.

$\triangle WAX$ and $\triangle YAZ$ both have height $h = \frac{45}{\sqrt{106}}$ and base $\frac{\sqrt{106}}{5}$, so
area $\triangle WAX$ = area $\triangle YAZ = \frac{1}{2}\left(\frac{\sqrt{106}}{5}\right)\left(\frac{45}{\sqrt{106}}\right) = \frac{9}{2}$ units^2.

Similarly, $\triangle WCX$ and $\triangle YCZ$ both have height $h = \frac{45}{\sqrt{106}}$ and base $\frac{\sqrt{106}}{5}$, so
area $\triangle WCX$ = area $\triangle YCZ = \frac{1}{2}\left(\frac{\sqrt{106}}{5}\right)\left(\frac{45}{\sqrt{106}}\right) = \frac{9}{2}$ units^2.

Therefore, the area of the shaded region is $4\left(\frac{9}{2}\right) = 18$ units^2.