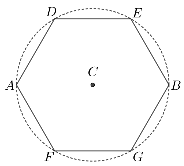
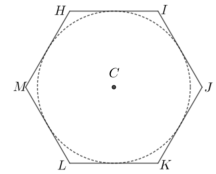


Problem of the Week
Problem D and Solution
Pi Day Hexagons



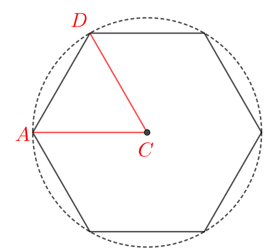
Problem

Archimedes determined lower and upper bounds for π by finding the perimeters of regular polygons inscribed and circumscribed in a circle with a diameter of length 1. We will determine such bounds by looking at regular hexagons inscribed and circumscribed in a circle with centre C and diameter 1. Since the circle has circumference equal to π, the perimeter of the inscribed regular hexagon $DEBGFA$ will give a lower bound of π and the perimeter of the circumscribed regular hexagon $HIJKLM$ will give an upper bound of π. Using these hexagons, determine a lower and an upper bound for π.

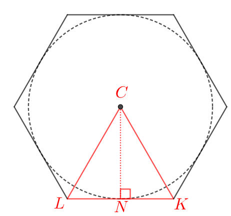
Solution

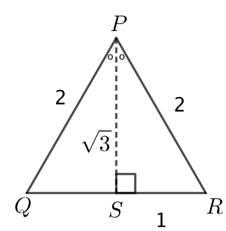
Solution 1

For the inscribed hexagon, draw line segments AC and DC, which are both radii of the circle. Since the diameter of the circle is 1, $AC = DC = \frac{1}{2}$. Since the inscribed hexagon is a regular hexagon with centre C, we know that $\triangle ACD$ is equilateral (a justification of this is provided at the end of the solution). Thus, $AD = AC = \frac{1}{2}$, and the perimeter of the inscribed regular hexagon is $6 \times AD = 6\left(\frac{1}{2}\right) = 3$. This gives us a lower bound for π.



For the circumscribed hexagon, draw line segments LC and KC. Since the circumscribed hexagon is a regular hexagon with centre C, we know that $\triangle LCK$ is equilateral (a justification of this is provided at the end of the solution). Drop a perpendicular from C, meeting LK at N. Since CN is a radius of the circle, $CN = \frac{1}{2}$.





Consider $\triangle PQR$ above, which is an equilateral triangle with side length of 2. Drop a perpendicular from P, meeting QR at S. Now, $\triangle PSR$ is similar to $\triangle CNK$ since $\angle PRS = \angle CKN = 60^\circ$, $\angle PSR = \angle CNK = 90^\circ$ and $\angle SPR = \angle NCK = \frac{1}{2}(60^\circ) = 30^\circ$.

Therefore,
$$\begin{aligned} \frac{CN}{PS} &= \frac{CK}{PR} \\ \frac{\frac{1}{2}}{\sqrt{3}} &= \frac{CK}{2} \\ \frac{1}{\sqrt{3}} &= CK \end{aligned}$$

Since $\triangle LCK$ is equilateral, $LK = CK = \frac{1}{\sqrt{3}}$. Thus, the perimeter of the regular hexagon is $6 \times LK = 6\left(\frac{1}{\sqrt{3}}\right) = \frac{6}{\sqrt{3}} \approx 3.46$. This gives us an upper bound for π.

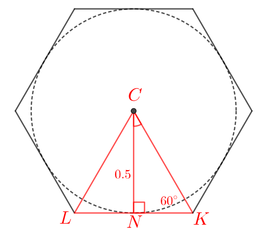
Therefore, the value for π is between 3 and $\frac{6}{\sqrt{3}} \approx 3.46$. That is, $3 < \pi < \frac{6}{\sqrt{3}}$.

Solution 2

This solution uses trigonometry.

The calculation of the perimeter of the inscribed hexagon is the same as in Solution 1.

For the circumscribed hexagon, draw line segments LC and KC. Since the circumscribed hexagon is a regular hexagon with centre C, we know that $\triangle LCK$ is equilateral (a justification of this is provided at the end of the solution). Thus, $\angle LKC = 60°$.



Drop a perpendicular from C, meeting LK at N. In $\triangle CNK$, $\angle NKC = \angle LKC = 60°$. Also, CN is a radius of the circle, so $CN = 0.5$. Since $\angle CNK = 90°$,

$$\begin{aligned} \sin(\angle NKC) &= \frac{CN}{KC} \\ \sin(60°) &= \frac{0.5}{KC} \\ \frac{\sqrt{3}}{2} &= \frac{0.5}{KC} \\ \sqrt{3}KC &= 1 \\ KC &= \frac{1}{\sqrt{3}} \end{aligned}$$

But $\triangle LCK$ is equilateral so $LK = KC = \frac{1}{\sqrt{3}}$.

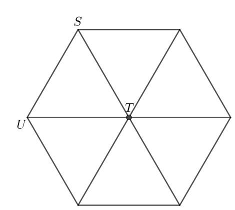
Thus, the perimeter of the circumscribed hexagon is $6 \times LK = 6 \times \frac{1}{\sqrt{3}} = \frac{6}{\sqrt{3}} \approx 3.46$. This gives us an upper bound for π.

Therefore, the value for π is between 3 and $\frac{6}{\sqrt{3}}$. That is, $3 < \pi < \frac{6}{\sqrt{3}}$.

EXTENSION: Archimedes used regular 12-gons, 24-gons, 48-gons and 96-gons to get better approximations for the bounds on π. Can you?

Equilateral triangle justification:
In the solutions, we used the fact that both $\triangle ACD$ and $\triangle LCK$ are equilateral. In fact, a regular hexagon can be split into six equilateral triangles by drawing line segments from the centre of the hexagon to each vertex, which we will now justify.

Consider a regular hexagon with centre T. Draw line segments from T to each vertex. Since T is the centre of the hexagon, T is of equal distance to each vertex of the hexagon. Since the hexagon is a regular hexagon, each side of the hexagon has equal length. Thus, the six resultant triangles are congruent. Therefore, the six central angles are equal and each is equal to $\frac{1}{6}(360°) = 60°$.



Now consider $\triangle STU$. We know that $\angle STU = 60°$. Also, $ST = UT$, so $\triangle STU$ is isosceles and $\angle TSU = \angle TUS = \frac{180° - 60°}{2} = 60°$. Therefore, all three angles in $\triangle STU$ are equal to $60°$ and $\triangle STU$ is equilateral. Since the six triangles in the hexagon are congruent, this tells us that the six triangles are all equilateral.