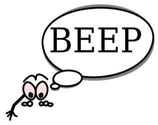

Problem of the Week
Problem D and Solution
Beep Beep

Problem

The game *Beep* is played by a group of people counting up through the positive integers from 1. The first person says "one", the second "two", and so on. However, every time a multiple of 9, or a number containing the digit 9 is encountered, to avoid losing, the person must say "beep" instead of stating the number. For example, one part of the game would sound like this: "twelve", "thirteen", "fourteen", "fifteen", "sixteen", "seventeen", "beep", "beep", "twenty". What number would they need to make it to in order to have heard "beep" exactly 300 times?

Solution

We first determine the number of integers from 1 to 100 that are replaced by a "beep".
Since $100 = (11 \times 9) + 1$, there are 11 multiples of 9 between 1 and 100.
The integers from 1 to 100 that contain the digit 9 are $9, 19, \ldots, 79, 89$ as well as $90, 91, \ldots, 97, 98, 99$. Thus, there are 19 positive integers from 1 to 100 that contain the digit 9.
Some integers that are multiples of 9 will also contain the digit 9, and will have been counted twice. There are 3 integers from 1 to 100 that are multiples of 9 (they are 9, 90, and 99).
Hence, the number of integers from 1 to 100 replaced by a "beep" is $11 + 19 - 3 = 27$.

We now determine the number of integers from 101 to 899 that are replaced by a "beep".
Since $899 = (99 \times 9) + 8$, there are 99 multiples of 9 between 1 and 899. Since there are 11 multiples of 9 between 1 and 100, there are $99 - 11 = 88$ multiples of 9 between 101 and 899.
Also, since there are 19 integers from 1 to 100 that contain the digit 9, there are $19 \times 8 = 152$ integers from 101 to 899 that contain the digit 9.
We now determine the number of integers between 101 and 899 that are both multiples of 9 and contain the digit 9, and thus have been counted twice.
For an integer from 101 to 899 to contain the digit 9, either the second or third digit is 9.
Therefore, these numbers are of the form $A9B$ or $AB9$, where $1 \leq A \leq 8$.
For the number to be a multiple of 9, $A + B + 9$ must equal $9, 18$ or 27. That is, $A + B$ is equal to $0, 9$ or 18.
If $A + B = 0$, then $A = 0$ and $B = 0$, which does not give an integer from 101 to 899.
If $A + B = 18$, then $A = 9$ and $B = 9$, which again does not give an integer from 101 to 899.
Therefore, we need to solve $A + B = 9$. The ordered pairs (A, B) which satisfy $A + B = 9$, $1 \leq A \leq 8$, and $0 \leq B \leq 9$ are $(1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1)$.
Each of these ordered pairs gives two different three-digit integers that are multiples of 9. For example, the ordered pair $(1, 8)$ gives the numbers 198 and 189. Therefore, from 101 to 899, there are $2 \times 8 = 16$ integers that are multiples of 9 and contain the digit 9.
Hence, the number of integers replaced by a "beep" from 101 to 899 is $88 + 152 - 16 = 224$.

Thus, from 1 to 899 there will be $27 + 224 = 251$ integers replaced by a "beep".
Therefore, we need "beep" to be said $300 - 251 = 49$ more times. Beginning at 900, every integer up to 999 contains the digit 9, and thus will be replaced with a "beep". The 49^{th} integer in this range is 948.

In conclusion, when the number 948 is reached they will have heard "beep" exactly 300 times.