Problem of the Week
Problem D and Solution
Adding Digits

Problem

ABC is a three-digit integer whose first digit is A, second digit is B and third digit is C. Similarly, DEF is a three-digit integer whose first digit is D, second digit is E and third digit is F. We are given:

\[
\begin{array}{c}
A & B & C \\
+ & D & E & F \\
\hline
1 & 2 & 3 & 4
\end{array}
\]

How many different values of \(A + B + C + D + E + F\) are there?

Solution

To solve this problem, we are going to look at each column starting with the units, then tens, and then finally the hundreds column.

Since \(C + F\) ends in a 4, then \(C + F = 4\) or \(C + F = 14\). The value of \(C + F\) cannot be 20 or more, as \(C\) and \(F\) are digits. In the case that \(C + F = 14\), we “carry” a 1 to the tens column.

Since the result in the tens column is 3, then when there is no “carry” from the units column, \(B + E\) ends in a 3, and when there is a “carry” from the units column \(1 + B + E\) ends in a 3, so \(B + E\) ends in a 2.

If \(B + E\) ends in a 3, then \(B + E = 3\) or \(B + E = 13\). The value of \(B + E\) cannot be 20 or more, as \(B\) and \(E\) are digits. In the case that \(B + E = 13\), we “carry” a 1 to the hundreds column.

If \(B + E\) ends in a 2, then \(B + E = 2\) or \(B + E = 12\). The value of \(B + E\) cannot be 20 or more, as \(B\) and \(E\) are digits. In the case that \(B + E = 12\), we “carry” a 1 to the hundreds column.

Since the result in the hundreds column is 12, then \(A + D = 12\), or in the case when there was a “carry” from the tens column \(1 + A + D = 12\), so \(A + D = 11\).

We summarize this information in a tree.

\[
\begin{array}{c}
Possible Sums \\
C + F = 4 & \quad C + F = 14 \\
| & | \\
B + E = 3 & B + E = 13 & B + E = 2 & B + E = 12 \\
\mid & \mid & \mid & \mid \\
A + D = 12 & A + D = 11 & A + D = 12 & A + D = 11
\end{array}
\]

The branches above give all possible sums.

The first branch has the sum \(A + B + C + D + E + F = 4 + 3 + 12 = 19\).

The second branch has the sum \(A + B + C + D + E + F = 4 + 13 + 11 = 28\).

The third branch has the sum \(A + B + C + D + E + F = 14 + 2 + 12 = 28\).

The fourth branch has the sum \(A + B + C + D + E + F = 14 + 12 + 11 = 37\).

Therefore there are 3 different sums for \(A + B + C + D + E + F\). They are 19, 28, and 37.
Indeed, we can find values for $A, B, C, D, E, F$ that achieve each of these sums.

When $A = 9, B = 2, C = 4, D = 3, E = 1, F = 0$, $A + B + C + D + E + F = 19$ and $ABC + DEF = 924 + 310 = 1234$.

When $A = 7, B = 9, C = 2, D = 4, E = 4, F = 2$, $A + B + C + D + E + F = 28$ and $ABC + DEF = 792 + 442 = 1234$.

When $A = 3, B = 5, C = 8, D = 8, E = 7, F = 6$, $A + B + C + D + E + F = 37$ and $ABC + DEF = 358 + 876 = 1234$. 