Problem of the Week
Problem D and Solution
Different Dice, Same Outcome

Problem

For mathematicians, a fair die means there is an equally likely chance of landing on any face of the die. A standard six-sided die has its faces marked with the numbers 1, 2, 3, 4, 5, and 6. The die is fair and each number is used exactly once. A special six-sided die has its faces marked with the numbers 1, 3, 4, 5, 6, and 8. The die is fair and each number is used exactly once. Is it possible to create a second fair, six-sided die marked so that this die and the first special die can be used together to play a board game like *Monopoly*? (Numbers on the faces of this new die may appear more than once.) In other words, does another fair, six-sided die exist so that when this new die and the special die are thrown the sum of the numbers on the top faces of the two dice range from 2 to 12 and the probability of obtaining each sum is the same as it would be if two standard dice had been thrown? If it is possible, what numbers would be on the faces of this new die? If it is not possible, explain why not.

Solution

We first examine what happens when two standard dice are thrown. For each possible roll, we calculate the sum of numbers on the top faces of the dice, and compile this information in a table.

Die 1 \ Die 2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

From this table we can determine the probability of each sum by counting the number of ways to get that sum and dividing by 36, the size of the sample space.

$$P(\text{total} = 2) = P(\text{total} = 12) = \tfrac{1}{36} \qquad P(\text{total} = 3) = P(\text{total} = 11) = \tfrac{2}{36}$$
$$P(\text{total} = 4) = P(\text{total} = 10) = \tfrac{3}{36} \qquad P(\text{total} = 5) = P(\text{total} = 9) = \tfrac{4}{36}$$
$$P(\text{total} = 6) = P(\text{total} = 8) = \tfrac{5}{36} \qquad P(\text{total} = 7) = \tfrac{6}{36}$$

We need our new die and the special die to give these same probabilities when rolled together. Since both the special and the new die have 6 sides, the size of the sample space will still be $6 \times 6 = 36$.

We need the sums of the top faces to range from 2 to 12. Since the numbers on our special die are 1, 3, 4, 5, 6 and 8, this means that on our new die, the smallest number must be 1 and the largest number must be 4. Now, let's examine the particular probabilities.

We need the probability that a sum of 2 is thrown to be $\frac{1}{36}$. Since the size of the sample space is 36, this means that we need exactly 1 way to get a sum of 2. Therefore, exactly one face on our new die must have the number 1 on it.

We need the probability that a sum of 12 is thrown to be $\frac{1}{36}$. Again, since the size of the sample space is 36, this means that we need exactly 1 way to get a sum of 12. Therefore, exactly one face on the new die must have the number 4 on it.

Therefore, on the new die the numbers on the six faces range from 1 to 4, and exactly one face has a 1 on it and exactly one face has a 4 on it. Therefore, there must be only 2's and 3's on the remaining 4 faces.

We need the probability that a sum of 3 is thrown to be $\frac{2}{36}$. Since the size of the sample space is 36, this means that there must be exactly 2 ways to get a sum of 3. If our new die has one face with a 2 on it, then there would be one way to get a sum of three: by throwing a 1 on the special die and a 2 on our new die. Therefore, we would need a second 2 on the new die. If there was third face with a 2, then there would be a third way to roll a sum of 3, which is too many.

Similarly, we need the probability that a sum of 11 is thrown to be $\frac{2}{36}$. Since the size of the sample space is 36, this means that there must be exactly 2 ways to get a sum of 11. If our new die has one face with a 3 on it, then there would be one way to get a sum of eleven: by throwing an 8 on the special die and a 3 on our new die. Therefore, we would need a second 3 on the new die. If there was a third face with a 3, then there would be a third way to roll a sum of 11, which is too many. Therefore, the faces on the new die, if it exists, will be 1, 2, 2, 3, 3, 4.

Let's check that this new die satisfies the conditions of the problem. For each possible roll with the special die and this new die, we calculate the sum of the numbers on the top faces of the dice and compile this information in a table.

Special Die \ New Die	1	2	2	3	3	4
1	2	3	3	4	4	5
3	4	5	5	6	6	7
4	5	6	6	7	7	8
5	6	7	7	8	8	9
6	7	8	8	9	9	10
8	9	10	10	11	11	12

From this table, we can determine the probability of different sums. For each possible sum, we count the number of ways to get that sum, and divide by 36, the size of the sample space.

$\text{P(total} = 2) = \text{P(total} = 12) = \frac{1}{36}$ $\text{P(total} = 3) = \text{P(total} = 11) = \frac{2}{36}$

$\text{P(total} = 4) = \text{P(total} = 10) = \frac{3}{36}$ $\text{P(total} = 5) = \text{P(total} = 9) = \frac{4}{36}$

$\text{P(total} = 6) = \text{P(total} = 8) = \frac{5}{36}$ $\text{P(total} = 7) = \frac{6}{36}$

Therefore, a second die exists with faces numbered 1, 2, 2, 3, 3, and 4, such that when this die and the special die are thrown, the probability of obtaining each sum is the same as it would be if two standard dice had been thrown. The existence of this die may be surprising. Can you find another pair of special dice that can do the same thing?