Problem of the Week
Problem D and Solution
New Year Sum

Problem

The positive integers are written consecutively in groups of seven so that the first row contains the numbers 1, 2, 3, 4, 5, 6, 7; the second row contains the numbers 8, 9, 10, 11, 12, 13, 14; the third row contains the numbers 15, 16, 17, 18, 19, 20, 21; etc. The *row sum* of a row is the sum of the numbers in the row. For example, the row sum of the first row is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$. Determine the numbers in the row that has a row sum closest to 2 019.

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Solution

Solution 1

Observe that the last number in any row is a multiple of 7. If n is the row number, then the last number in the n^{th} row is $7n$. Since the last number in row n is $7n$, the six preceding numbers in the row are $7n - 1$, $7n - 2$, $7n - 3$, $7n - 4$, $7n - 5$, and $7n - 6$.

The sum of the numbers in the n^{th} row is

$$(7n - 6) + (7n - 5) + (7n - 4) + (7n - 3) + (7n - 2) + (7n - 1) + 7n$$

which simplifies to $49n - 21$. We want to find the integer value of n so that $49n - 21$ is as close to 2 019 as possible.

$$\begin{aligned} 49n - 21 &= 2\,019 \\ 49n &= 2\,040 \\ n &\approx 41.6 \end{aligned}$$

The closest integer to 41.6 is 42. Therefore, $n = 42$ and the row sum is $49n - 21 = 49(42) - 21 = 2037$. The last number in the 42^{nd} row is $7 \times 42 = 294$. The seven numbers in the 42^{nd} row are 288, 289, 290, 291, 292, 293 and 294. The 41^{st} row contains the numbers 281, 282, 283, 284, 285, 286 and 287, and the row sum is 1 988. This row sum is farther from 2 019 than the 42^{nd} row sum of 2037 is.

Therefore, the row with the sum closest to 2 019 contains the numbers 288, 289, 290, 291, 292, 293 and 294.

The second solution approaches the problem by establishing a linear relationship.

Solution 2

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Let x represent the row number and y represent the sum of the numbers in the row. Observe that the seventh number in any row is a multiple of 7. In fact, the seventh number in any row is 7 times the row number or $7x$. The following table of values shows the row sums for the first three rows.

Row Number x	Row Sum y
1	28
2	77
3	126

It appears that the y values increase by 49 as the x values increase by 1. We will verify that this is true. In solution 1 we saw that the sum of the numbers in the n^{th} row is $49n - 21$. Therefore, the sum of the numbers in the x^{th} row is $49x - 21$ and the sum of the numbers in the $(x+1)^{\text{th}}$ row is $49(x+1) - 21 = (49x - 21) + 49$. Thus, the y values increase by 49 as the x values increase by 1. This tells us that the sum of the fourth row should be $126 + 49 = 175$. We can verify this by adding $22 + 23 + 24 + 25 + 26 + 27 + 28$, the numbers in the fourth row. The sum is indeed 175.

As the values of x increase by 1, the values of y increase by 49. The relation is linear. The slope is $\dfrac{\Delta y}{\Delta x} = \dfrac{49}{1} = 49$. Substituting $x = 1,\ y = 28,\ m = 49$ into

$$\begin{aligned} y &= mx + b \\ 28 &= 49(1) + b \\ -21 &= b \end{aligned}$$

The equation of the line which passes through the points in the relation is $y = 49x - 21$. Note that x and y are positive integers. We want to find the value of x, the row number, so that the value of y, the row sum, is as close to 2 019 as possible.

$$\begin{aligned} 49x - 21 &= 2\,019 \\ 49x &= 2\,040 \\ x &= 41.6 \end{aligned}$$

We want the integer value for x that is closest to 41.6. Therefore, $x = 42$ and the row sum is $y = 49(42) - 21 = 2037$. The row sum when $x = 41$ is $y = 49(41) - 21 = 1988$. The row sum 2037 is closer to 2019 than the row sum 1988.

The seventh number in the 42^{nd} row is $7 \times 42 = 294$. The seven numbers in the 42^{nd} row are 288, 289, 290, 291, 292, 293, and 294.

Therefore, the row with the sum closest to 2 019 contains the numbers 288, 289, 290, 291, 292, 293, and 294, and the row sum is 2037.