Problem of the Week
Problem E and Solution
Group of Six

Problem

ABC is a three-digit integer whose first digit is A, second digit is B, and third digit is C. Similarly, DEF is a three-digit integer whose first digit is D, second digit is E, and third digit is F. We are given that

\[
\begin{array}{c}
A B C \\
+ D E F \\
\hline
1 2 3 4
\end{array}
\]

How many 6-tuples \((A, B, C, D, E, F)\) are there with \(A > D\), \(B > E\), and \(C > F\) that make the above statement true?

Solution

To solve this problem, we are going to look at each column starting with the units, then tens, and then finally the hundreds column.

Since \(C + F\) ends in a 4, then \(C + F = 4\) or \(C + F = 14\). The value of \(C + F\) cannot be 20 or more, as \(C\) and \(F\) are digits. In the case that \(C + F = 14\), we “carry” a 1 to the tens column.

Since the result in the tens column is 3, then when there is no “carry” from the units column, \(B + E\) ends in a 3, and when there is a “carry” from the units column \(1 + B + E\) ends in a 3, so \(B + E\) ends in 2.

If \(B + E\) ends in a 3, then \(B + E = 3\) or \(B + E = 13\). The value of \(B + E\) cannot be 20 or more, as \(B\) and \(E\) are digits. In the case that \(B + E = 13\), we “carry” a 1 to the hundreds column.

If \(B + E\) ends in a 2, then \(B + E = 2\) or \(B + E = 12\). The value of \(B + E\) cannot be 20 or more, as \(B\) and \(E\) are digits. In the case that \(B + E = 12\), we “carry” a 1 to the hundreds column.

Since the result in the hundreds column is 12, then \(A + D = 12\), or in the case when there was a “carry” from the tens column \(1 + A + D = 12\), so \(A + D = 11\).

We summarize this information in a tree.

Possible Sums

\[
\begin{array}{c}
C + F = 4 \\
| \\
B + E = 3 \\
| \\
A + D = 12 \\
\hline
| \\
C + F = 14 \\
| \\
B + E = 2 \\
| \\
A + D = 12 \\
\hline
| \\
B + E = 13 \\
| \\
A + D = 11 \\
\hline
| \\
B + E = 12 \\
| \\
A + D = 11
\end{array}
\]
We now look at the different possibilities for digits $A, B, C, D, E,$ and $F$ for each individual sum, with the restriction that $A > D$, $B > E$, and $C > F$.

<table>
<thead>
<tr>
<th>Sum</th>
<th>Possible solutions</th>
<th>Total Number of Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C + F = 4$</td>
<td>$C = 4, F = 0$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$C = 3, F = 1$</td>
<td></td>
</tr>
<tr>
<td>$C + F = 14$</td>
<td>$C = 9, F = 5$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$C = 8, F = 6$</td>
<td></td>
</tr>
<tr>
<td>$B + E = 2$</td>
<td>$B = 2, E = 0$</td>
<td>1</td>
</tr>
<tr>
<td>$B + E = 3$</td>
<td>$B = 3, E = 0$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$B = 2, E = 1$</td>
<td></td>
</tr>
<tr>
<td>$B + E = 12$</td>
<td>$B = 9, E = 3$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$B = 8, E = 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$B = 7, E = 5$</td>
<td></td>
</tr>
<tr>
<td>$B + E = 13$</td>
<td>$B = 9, E = 4$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$B = 8, E = 5$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$B = 7, E = 6$</td>
<td></td>
</tr>
<tr>
<td>$A + D = 11$</td>
<td>$A = 9, D = 2$</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>$A = 8, D = 3$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A = 7, D = 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A = 6, D = 5$</td>
<td></td>
</tr>
<tr>
<td>$A + D = 12$</td>
<td>$A = 9, D = 3$</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$A = 8, D = 4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$A = 7, D = 5$</td>
<td></td>
</tr>
</tbody>
</table>

We can find the number of 6-tuples that are possible for the first branch of the tree. We have 2 choices for the units column. For each of these two choices, we have 2 choices for the tens column, so there are $2 \times 2 = 4$ possibilities. For each of these possibilities, we have 3 choices for the hundreds column. So the number of 6-tuples is $2 \times 2 \times 3 = 12$.

Similarly, for the second branch, we have 2 choices for the units column, 3 choices for the tens column, and 4 choices for the hundreds column. So the number of 6-tuples is $2 \times 3 \times 4 = 24$.

For the third branch, we have 2 choices for the units column, 1 choice for the tens column, and 3 choices for the hundreds column. So the number of 6-tuples is $2 \times 1 \times 3 = 6$.

For the fourth branch, we have 2 choices for the units column, 3 choices for the tens column, and 4 choices for the hundreds column. So the number of 6-tuples is $2 \times 3 \times 4 = 24$.

Therefore, the total number of 6-tuples is $12 + 24 + 6 + 24 = 66$. 