Problem of the Week
Problem E and Solution
An Uphill Struggle

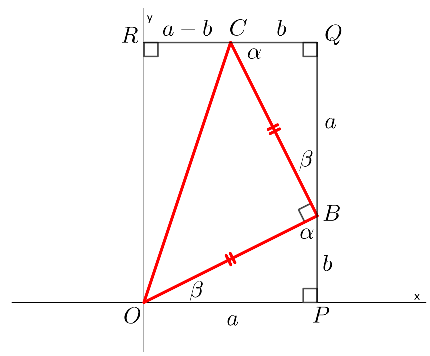
Problem

The following information is known about $\triangle OBC$: O is at the origin and points B and C lie in the first quadrant; $\triangle OBC$ is an isosceles right triangle with $OB = BC$ and $\angle OBC = 90^\circ$; and the hypotenuse OC is on a line segment with slope 3. Determine the slope of line segment OB.

Solution

We present three solutions. The first involves a construction. The second solution follows after making an assumption. The third solution uses trigonometry. The formula used in the third solution may not be familiar to all students.

Solution 1



Draw a line through C parallel to the x-axis, intersecting the y-axis at R.

Draw a line through B parallel to the y-axis, intersecting the x-axis at P and intersecting the first line through R and C at Q.

This construction creates rectangle $OPQR$.

In $\triangle CQB$, let $\angle QCB = \alpha$ and $\angle QBC = \beta$. Since $OPQR$ is a rectangle, $\angle BQC = 90^\circ$ and $\triangle CQB$ is a right angled triangle. It follows that $\alpha + \beta = 90^\circ$.

$\angle QBP$ is a straight angle so $\angle QBC + \angle CBO + \angle OBP = 180^\circ$.
Substituting, we obtain $\beta + 90^\circ + \angle OBP = 180^\circ$ which simplifies to $\beta + \angle OBP = 90^\circ$.
But $\alpha + \beta = 90^\circ$ so it follows that $\angle OBP = \alpha$. Then in right triangle BPO, we get $\angle BOP = \beta$.

In $\triangle CQB$ and $\triangle BPO$, since $\angle QCB = \angle OBP = \alpha$, $\angle QBC = \angle BOP = \beta$, and $BC = OB$ (given), then $\triangle CQB \cong \triangle BPO$.

From the triangle congruence, we get $CQ = BP = b$ and $QB = OP = a$.

In rectangle $OPQR$, $RC + CQ = OP$. Substituting, we obtain $RC + b = a$ and $RC = a - b$ follows.

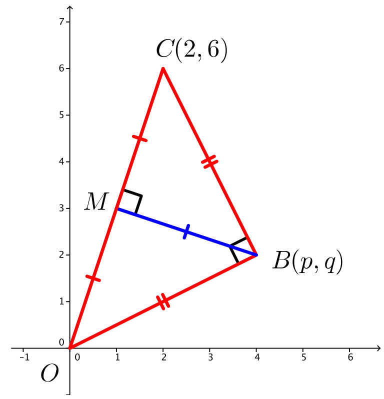
All of this information is shown on the diagram above.

The coordinates of C are $(a - b, a + b)$ and the coordinates of B are (a, b).

We know the slope of $OC = 3$, so $\dfrac{a+b}{a-b} = 3$. Simplifying, we obtain $a + b = 3a - 3b$ and $a = 2b$ follows.

Then the slope of $OB = \dfrac{b}{a} = \dfrac{b}{2b} = \dfrac{1}{2}$.

Solution 2



Since OC is a line segment with slope 3, with O at the origin and C in the first quadrant, the coordinates of C will be of the form $(a, 3a)$, where a is some positive number. We will do our calculations with $a = 2$. Then the length of OC is $2\sqrt{10}$. Let B be the point (p, q).

Let M be the midpoint of OC. Then M is the point $(1, 3)$. It follows that $OM = MC = \frac{1}{2}OC = \sqrt{10}$.

In an isosceles right triangle, the line segment joining the midpoint of the hypotenuse to the opposite vertex is perpendicular to the hypotenuse and has length equal to half the length of the hypotenuse. (If this result is not known, it is easily shown using congruent triangles.)
It follows that $MB \perp OC$ and $MB = \sqrt{10}$.

Since $MB \perp OC$ and the slope of OC is 3, then the slope of MB is $-\frac{1}{3}$. We can find the equation of the line containing $M(1, 3)$ with slope $-\frac{1}{3}$ by substituting into $y = mx + b$.

$$\begin{aligned} 3 &= -\frac{1}{3}(1) + b \\ 9 &= -1 + 3b \\ 10 &= 3b \\ \frac{10}{3} &= b \end{aligned}$$

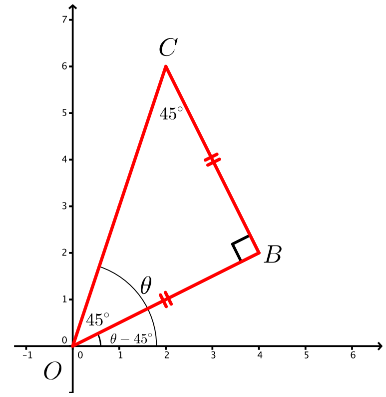
The equation of the line containing MB is $y = -\frac{1}{3}x + \frac{10}{3}$.

Since $B(p, q)$ is on this line, $q = -\frac{1}{3}p + \frac{10}{3}$. (1)

The length of MB is $\sqrt{10}$. Using $M(1, 3)$ and $B(p, -\frac{1}{3}p + \frac{10}{3})$,

$$\begin{aligned} MB^2 &= (p-1)^2 + \left(-\frac{1}{3}p + \frac{10}{3} - 3\right)^2 \\ (\sqrt{10})^2 &= (p-1)^2 + \left(-\frac{1}{3}p + \frac{1}{3}\right)^2 \\ 10 &= (p-1)^2 + \left(-\frac{1}{3}(p-1)\right)^2 \\ 10 &= (p-1)^2 + \frac{1}{9}(p-1)^2 \\ 10 &= \frac{10}{9}(p-1)^2 \\ 9 &= (p-1)^2 \\ \pm 3 &= p - 1 \end{aligned}$$

It follows that $p = 4$ or $p = -2$. Since B is in quadrant 1, $p = -2$ is inadmissible. Therefore, $p = 4$. Substituting in (1), $q = 2$ and B is the point $(4, 2)$. Thus, the slope of $OB = \frac{2}{4} = \frac{1}{2}$.



Solution 3

Since $\triangle OBC$ is an isosceles right triangle with $\angle OBC = 90^\circ$, then $\angle BOC = \angle BCO = 45^\circ$.

Let θ represent the angle that OC makes with the positive x-axis. Since the slope of $OC = 3$, then $\tan\theta = 3$, since the tangent of an angle is equal to the slope of a line that makes this angle with the horizontal (the positive x-axis in this case).

The angle that OB makes with the positive x-axis is $\theta - 45^\circ$. The slope of OB will equal $\tan(\theta - 45^\circ)$.

Using the fact that $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$,

$$\begin{aligned}\tan(\theta - 45^\circ) &= \frac{\tan\theta - \tan 45^\circ}{1 + \tan\theta\tan 45^\circ}\\ &= \frac{3-1}{1+3(1)}\\ &= \frac{2}{4}\\ &= \frac{1}{2}\end{aligned}$$

Therefore, the slope of $OB = \tan(\theta - 45^\circ) = \dfrac{1}{2}$.