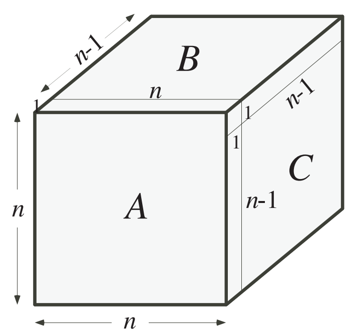
Problem of the Week

Problem E and Solution

Paint Free

Problem

A cube has edges of length n, where n is a positive integer. Three faces, meeting at a corner, are painted red. The cube is then cut into n^3 smaller cubes of unit length. That is, the side lengths of the new cubes are 1 unit. If exactly 125 of these cubes have no faces painted red, determine the value of n.



Solution

Solution 1

This solution requires no content beyond grade ten. The second solution will use the factor theorem which is generally taught in grade twelve.

In the diagram above, the sides painted red are labelled A, B, and C. We know that there are n^3 unit cubes. To determine the number of unpainted cubes we can subtract the number of cubes with some red from the total number of cubes. Side A has dimensions n by n by 1 and so contains n^2 unit cubes with some red. Side B has dimensions n by $(n-1)$ by 1 and so contains $n \times (n-1)$ unit cubes with some red. Side C has dimensions $(n-1)$ by $(n-1)$ by 1 and so contains $(n-1)(n-1)$ unit cubes with some red. The number of unpainted cubes is $n^3 - n^2 - n(n-1) - (n-1)(n-1)$. We can simplify this as follows:

$$n^3 - n^2 - n(n-1) - (n-1)(n-1) = n^2(n-1) - n(n-1) - (n-1)(n-1)$$

Each term contains a common factor of $(n-1)$ so the expression simplifies to $(n-1)(n^2 - n - (n-1)) = (n-1)(n^2 - 2n + 1)$. This further simplifies to $(n-1)^3$. If the solver pauses here to think about this, if the unit cubes on side A then side B and finally side C are removed we are left with a cube whose side lengths are $(n-1)$ and $(n-1)^3$ unit cubes.

But $(n-1)^3 = 125$, the actual number of unpainted cubes. Taking the cube root, $n - 1 = 5$ and $n = 6$ follows.

Solution 2

The second solution will use the factor theorem which is generally taught in grade twelve.

This solution picks up from the expression giving us the number of unpainted cubes, setting it equal to 125, the number of unpainted cubes.

$$\begin{aligned} n^3 - n^2 - n(n-1) - (n-1)(n-1) &= 125 \\ n^3 - n^2 - n^2 + n - n^2 + 2n - 1 &= 125 \\ n^3 - 3n^2 + 3n - 126 &= 0 \end{aligned}$$

Let $f(n) = n^3 - 3n^2 + 3n - 126$.

When $n = 6$, $f(6) = 6^3 - 3(6^2) + 3(6) - 126 = 216 - 108 + 18 - 126 = 224 - 224 = 0$. Since $f(6) = 0$, $(n-6)$ is a factor of $f(n)$.

After long division (or synthetic division), $f(n) = (n-6)(n^2 + 3n + 21)$.

So $(n-6)(n^2 + 3n + 21) = 0$. $n^2 + 3n + 21 = 0$ has no real roots so $n = 6$ is the only root.

Therefore the original cube has edges of length 6.