Problem of the Week
Problem E and Solution
Many Possibilities?

Problem

Two distinct positive integers are multiplied together. This product is then added to the sum of the original two integers resulting in a sum of 195.

Determine all possible pairs of integers which satisfy all of the conditions.

Solution

Let x and y represent the two positive integers. Since the integers are distinct, $x \neq y$.
The product of the two integers is xy and the sum is $(x + y)$.
We want to find all pairs of integers, $x,\ y$, such that $xy + x + y = 195$.

If we attempt to factor the left side of the equation, we discover that there is no common factor between all three terms. However, if we factor a common factor out of the first two terms on the left side of the equation, we obtain $x(y + 1) + y = 195$.

If we add 1 to both sides, we obtain:

$$x(y + 1) + y + 1 = 195 + 1$$
$$\text{or } x(y + 1) + 1(y + 1) = 196.$$

Now, the left side has a common factor of $(y + 1)$. After factoring, we obtain:

$$(x + 1)(y + 1) = 196.$$

Looking strictly at the equation $(x + 1)(y + 1) = 196$, we see that we are looking for a pair of integers whose product is 196. Using the factors of 196, we obtain

$$196 = 1 \times 196 = 2 \times 98 = 4 \times 49 = 7 \times 28 = 14 \times 14.$$

We could also list the first four products in reverse order but the pairs of numbers producing these products will be the same as the pairs producing the first four products already listed.

The numbers in the products are each one more than the numbers we are looking for.

For the product $196 = 1 \times 196$, $x = 0$ and $y = 195$. Since the required numbers are positive integers, this solution is inadmissible.
For the product $196 = 2 \times 98$, $x = 1$ and $y = 97$. This is a valid solution.
For the product $196 = 4 \times 49$, $x = 3$ and $y = 48$. This is a valid solution.
For the product $196 = 7 \times 28$, $x = 6$ and $y = 27$. This is a valid solution.
For the product $196 = 14 \times 14$, $x = 13$ and $y = 13$. Since the required numbers are distinct positive integers, this solution is inadmissible.

There are three pairs of distinct positive integers for which their product and their sum add to 195. The pairs are 1 and 97, 3 and 48, and 6 and 27. A verification is provided on the next page.

First Positive Integer	Second Positive Integer	Product	Sum	Product + Sum
1	97	97	98	$97 + 98 = 195$
3	48	144	51	$144 + 51 = 195$
6	27	162	33	$162 + 33 = 195$

Note: if the two positive integers did not need to be distinct, then $x = 13$ and $y = 13$ would also be a valid solution since $13 \times 13 = 169$, $13 + 13 = 26$ and $169 + 26 = 195$.

If the problem asked for integer solutions, the number of solutions would increase still further. It is important to pay attention to any and all restrictions stated or implied in a problem.