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Problem of the Week
Problem E and Solution
Arithmetic Tendencies

Problem

The mean, median and mode of the eight positive numbers $10, 2, 5, 2, 6, 4, 2, x$ are distinct. The mean, median and mode are calculated and then listed in order from smallest to largest. The differences between adjacent numbers in this new list are equal.
Determine all possible values of x.

Solution

Since there are at least three 2's, the mode will be 2 regardless the value of x.

The mean of the numbers is $\dfrac{10+2+5+2+6+4+2+x}{8} = \dfrac{x+31}{8}$.

The median of the numbers will depend on the value of x compared to the other numbers. Since there are eight numbers in the list, the median will be the average of the fourth and fifth numbers in the ordered list of numbers. We will break the problem into cases.

Case 1: $0 < x \le 2$ and the ordered list is $x, 2, 2, 2, 4, 5, 6, 10$.

The median is the average of the fourth and fifth numbers, or $\frac{2+4}{2} = 3$.

Since $x > 0$, the mean is $\dfrac{x+31}{8} > \dfrac{31}{8} > 3$. It follows that the mean is greater than the median. The ordered list of numbers is then $2, 3, \dfrac{x+31}{8}$.

Since the differences between adjacent numbers are equal, we have

$$\begin{aligned}
3-2 &= \frac{x+31}{8} - 3\\
1 &= \frac{x+31}{8} - 3\\
4 &= \frac{x+31}{8}\\
32 &= x+31\\
1 &= x
\end{aligned}$$

So $x = 1$ and the mode, median and mean are $2,\ 3,\ 4$, respectively. Each adjacent pair of numbers in the list differs by 1.

Case 2: $2 < x \le 5$, and the ordered list is $2, 2, 2, x, 4, 5, 6, 10$ or $2, 2, 2, 4, x, 5, 6, 10$.

In both lists, the fourth and fifth numbers are x and 4. It follows that the median for both lists is $\frac{x+4}{2}$. We also know that the mode is 2. We do not, however, know which is larger, the median or the mean, so we look at both cases.

Case 2a: Let the median be smaller than the mean. Then the ordered list is

$$2,\ \frac{x+4}{2},\ \frac{x+31}{8}$$

Since the differences between adjacent numbers are equal, we have

$$\begin{aligned}\frac{x+4}{2}-2&=\frac{x+31}{8}-\frac{x+4}{2}\\ 4x+16-16&=x+31-4x-16 \quad \text{(Multiply both sides of the equation by 8.)}\\ 7x&=15\\ x&=\frac{15}{7}\end{aligned}$$

When $x=\frac{15}{7}$ the mode is 2, the median becomes $\frac{\frac{15}{7}+4}{2}=\frac{43}{14}$ and the mean becomes $\frac{\frac{15}{7}+31}{8}=\frac{29}{7}$. Each adjacent pair of numbers in the list differs by $\frac{15}{14}$.

Case 2b: Let the mean be smaller than the median. Then the ordered list is

$$2,\ \frac{x+31}{8},\ \frac{x+4}{2}$$

Since the differences between adjacent numbers are equal, we have

$$\begin{aligned}\frac{x+31}{8}-2&=\frac{x+4}{2}-\frac{x+31}{8}\\ x+31-16&=4x+16-x-31 \quad \text{(Multiply both sides of the equation by 8.)}\\ 30&=2x\\ x&=15\end{aligned}$$

But $x\le 5$, so there is no value of x that satisfies the conditions in this case.

Case 3: $x>5$ and the first 5 numbers in the list, in numerical order, are $2,2,2,4,5$. Since the fourth number in the list is 4 and the fifth number is 5, then it follows that the median is $\frac{9}{2}$. The mode is 2. Since $x>5$, the mean is $\frac{x+31}{8}>\frac{36}{8}=\frac{9}{2}$. It follows that the mean is greater than the median. The ordered list of numbers is then $2,\frac{9}{2},\frac{x+31}{8}$.

Since the differences between adjacent numbers are equal, we have

$$\begin{aligned}\frac{9}{2}-2&=\frac{x+31}{8}-\frac{9}{2}\\ 7&=\frac{x+31}{8}\\ 56&=x+31\\ 25&=x\end{aligned}$$

In this case, $x>5$ and therefore $x=25$ is a valid solution.

When $x = 25$ the mode is 2, the median $\frac{9}{2}$ and the mean becomes $\frac{25+31}{8} = 7$. Each adjacent pair of numbers in the list differs by $\frac{5}{2}$.

Therefore, there are three values of x that satisfy the conditions of the problem.

When $x = 1$, the ordered list is $1, 2, 2, 2, 4, 5, 6, 10$. The mean is 4, the median is 3 and the mode is 2. When listed from smallest to largest, the three numbers are $2, 3, 4$ and the difference between adjacent terms is 1.

When $x = \frac{15}{7}$, the ordered list is $2, 2, 2, \frac{15}{7}, 4, 5, 6, 10$. The mean is $\frac{29}{7}$, the median is $\frac{43}{14}$ and the mode is 2. When listed from smallest to largest, the three numbers are $2, \frac{43}{14}, \frac{29}{7}$ and the difference between adjacent terms is $\frac{15}{14}$.

When $x = 25$, the ordered list is $2, 2, 2, 4, 5, 6, 10, 25$. The mean is 7, the median is $\frac{9}{2}$ and the mode is 2. When listed from smallest to largest, the three numbers are $2, \frac{9}{2}, 7$ and the difference between adjacent terms is $\frac{5}{2}$.

Note: $2, 3, 4$ and $2, \frac{43}{14}, \frac{29}{7}$ and $2, \frac{9}{2}, 7$ are all examples of *arithmetic sequences*. An arithmetic sequence is a list of numbers with the difference between adjacent numbers being constant.