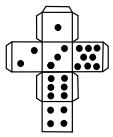

Problem of the Week
Problem E and Solution
Roll With It

Problem

A die, with the numbers 1, 2, 3, 4, 6, and 8 on its six faces, is rolled. (A net showing the six faces of the die is illustrated above.) If, after the first roll, the number appearing on the top face of the die is odd, then all of the odd numbers on the die are doubled. If, after the first roll, the number appearing on the top face of the die is even, then all of the even numbers on the die are halved. This new die is rolled. The rules stated above are applied to the outcome of this roll producing another new die. This second new die is then rolled. No change occurs after this roll. What is the probability that a 2 will be on the top face of the die after the third roll?

Solution

We will use the notation $(1, 2, 3, 4, 6, 8)$ to describe the original die.
For the first two rolls, we need to keep track of the parity only. For the third roll we need to need to look at an outcome of 2. There are therefore four possible cases for the first two rolls.

Case 1: First roll is even, second roll is even.
The probability of rolling an even on the first roll is $\frac{4}{6} = \frac{2}{3}$. The die will now be $(1, 1, 3, 2, 3, 4)$.
The probability of rolling an even on the second roll is $\frac{2}{6} = \frac{1}{3}$. The die will now be $(1, 1, 3, 1, 3, 2)$
The probability of rolling a 2 on the third roll is $\frac{1}{6}$.
The probability of rolling a 2 in this case is $\frac{2}{3} \times \frac{1}{3} \times \frac{1}{6} = \frac{1}{27}$.

Case 2: First roll is even, second roll is odd.
The probability of rolling an even on the first roll is $\frac{2}{3}$. The die will now be $(1, 1, 3, 2, 3, 4)$.
The probability of rolling an odd on the second roll is $\frac{2}{3}$. The die will now be $(2, 2, 6, 2, 6, 4)$
The probability of rolling a 2 on the third roll is $\frac{3}{6} = \frac{1}{2}$.
The probability of rolling a 2 in this case is $\frac{2}{3} \times \frac{2}{3} \times \frac{1}{2} = \frac{4}{18}$.

Case 3: First roll is odd, second roll is even.
The probability of rolling an odd on the first roll is $\frac{1}{3}$. The die will now be $(2, 2, 6, 4, 6, 8)$.
The probability of rolling an even on the second roll is 1. The die will now be $(1, 1, 3, 2, 3, 4)$
The probability of rolling a 2 on the third roll is $\frac{1}{6}$.
The probability of rolling a 2 in this case is $\frac{1}{3} \times 1 \times \frac{1}{6} = \frac{1}{18}$.

Case 4: First roll is odd, second roll is odd.
The probability of rolling an odd on the first roll is $\frac{1}{3}$. The die will now be $(2, 2, 6, 4, 6, 8)$.
The probability to rolling an odd on the second roll is 0.
The probability of getting a 2 in this case is 0.

The total probability is the sum of the probabilities of each case or $\frac{1}{27} + \frac{4}{18} + \frac{1}{18} + 0 = \frac{17}{54}$.

Therefore the probability of rolling a 2 on the third roll is $\frac{17}{54}$.